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ABSTRACT: Propositional modal logic of programs has been introduced by Fischer and Ladner [1], following ideas of Pratt [4]. We shall call it propositional dynamic logic (PDL) following the terminology of Harel, Meyer and Pratt. In the following we prove the completeness[1] of a rather natural set of axioms for this logic and for an extension of it obtained by allowing the inverse operation which converts a program into its inverse.

The following is a brief sketch of the plan of the proof. We introduce two auxiliary notions, that of pseudomodel and that of nonstandard model. Pseudomodels are highly syntactic objects and merely represent partial attempts to spell out a model. Thus an inconsistent formula may have a pseudomodel but every attempt to spell out the complete details of a model corresponding to the pseudomodel will, for an inconsistent formula, run into obstacles. A nonstandard model is like a model but we do not insist that $R_{\alpha*} = (R_\alpha)^*$. $R_{\alpha*}$ is some reflexive transitive relation containing R_α, but not necessarily the smallest.

We shall show that if a formula A is not disprovable from the axioms then it has a series of consistent pseudomodels whose union is a nonstandard model satisfying certain special induction axioms. It is then shown how such a nonstandard model can be converted into a model in the usual sense.

Definition 1: Consider the language of PDL whose nonlogical symbols consist of program letters $a_1,\ldots,a_n$ and propositional letters $P_1,\ldots,P_m$. We define the notion "formula" and "program" by simultaneous recursion.

1) Every program letter is a program.
2) Every propositional letter is a formula.
3) If A,B are formulae, so are $\neg A$, $(A \vee B)$.
4) If α,β are programs, so are α^*, α^{-1}, $\alpha \cup \beta$, $\alpha;\beta$.
5) If α is a program and A is a formula, then $[\alpha]A$ is a formula.
6) If A is a formula, A? is a program.

Remark: We shall follow the notational conventions implicit above. E.g., $a_1, a_2 \ldots$ for program letters, α,β,γ for arbitrary programs.

Definition 2: A PDL structure $\mathfrak{M}$, for the language above, consists of (1) a universe W, (2) for each program letter a, a binary relation $R_a \subseteq W \times W$. (3) for each $w \in W$ and each propositional letter P, a truth value

$v(w,P) \in \{T,F\}$. We then assign binary relations R_α to programs α and truth values to formulas by simultaneous recursion as follows:

1) $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$

2) $R_{\alpha;\beta} = R_\alpha \circ R_\beta = \{(w_1,w_2) \mid \exists w, (w_1,w) \in R_\alpha \wedge (w,w_2) \in R_\beta\}$

($\alpha;\alpha$ shall be abbreviated: α^2; similarly, α^n)

3) $R_{\alpha^*} = \bigcup_{n=0}^{\infty} R_{\alpha^n}$ 4) $R_{\alpha^{-1}} = (R_\alpha)^{-1} = \{(w_2,w_1) \mid (w_1,w_2) \in R_\alpha\}$

5) $\mathfrak{M},w \models P$ iff $v(w,P) = T$ 6) $\mathfrak{M},w \models \neg A$ iff $\mathfrak{M}, w \not\models A$

7) $\mathfrak{M},w \models A \vee B$ iff $\mathfrak{M},w \models A$ or $\mathfrak{M},w \models B$ 8) $R_{A?} = \{(w,w) \mid \mathfrak{M},w \models A\}$

9) $\mathfrak{M},w \models [\alpha]A$ iff $(\forall w')((w,w') \in R_\alpha \rightarrow \mathfrak{M},w' \models A)$.

A <u>nonstandard</u> structure is just as described above except that R_{α^*} is <u>some</u> reflexive transitive relation containing R_α. Condition 3 is dropped.

We shall consider $\rightarrow, \wedge, \leftrightarrow$ to be defined symbols in the usual way. $\langle\alpha\rangle A$ is an abbreviation for $\neg[\alpha]\neg A$. (Fischer and Ladner use $\langle\alpha\rangle$ as basic but we assume the reader can adapt.)

A formula A is <u>satisfiable</u> iff there are $\mathfrak{M}, w$ (<u>standard</u>) such that $\mathfrak{M},w \models A$. A is valid iff $\neg A$ is not satisfiable.

<u>Definition 3</u>: The axioms and rules of inference for PDL are as follows:

<u>Axioms</u>:

1) All tautologies. (Or enough of them.)
2) $[\alpha](A \rightarrow B) \rightarrow ([\alpha]A \rightarrow [\alpha]B)$
3) $[\alpha \cup \beta]A \leftrightarrow ([\alpha]A \wedge [\beta]A)$
4) $[\alpha;\beta]A \leftrightarrow [\alpha][\beta]A$
5) $[\alpha^*]A \rightarrow [\alpha]A$
6) $[\alpha^*]A \rightarrow A$
7) $[\alpha^*]A \rightarrow [\alpha^*][\alpha^*]A$
8) $A \rightarrow [\alpha]\langle\alpha^{-1}\rangle A$
9) $A \rightarrow [\alpha^{-1}]\langle\alpha\rangle A$
10) $A \wedge [\alpha^*](A \rightarrow [\alpha]A) \rightarrow [\alpha^*]A$
11) $[(\alpha^*)^{-1}]A \leftrightarrow [(\alpha^{-1})^*]A$.

It has been pointed out to us by Vaughan Pratt that axiom 11 is redundant. A proof of this fact is given in § 2. Richard Ladner points out that 5), 6), and 7) can be replaced by $[\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$. Axioms 10 will be called "induction axioms."

Rules:

(mp) $\dfrac{A,\ A \rightarrow B}{B}$ (gen) $\dfrac{A}{[\alpha]A}$.

Here A, B are arbitrary formulae, and α, β are arbitrary programs. If the axioms in which the -1 symbol appears are omitted, the resulting system is complete for $\cup$,;,*. Similarly if * is omitted.

Remark: $\vdash A$ will mean that A is a formal theorem, i.e. that it is provable in our formal system.

Theorem 1: A formula is valid iff it is provable from the axioms using the rules mp and gen.

The proof of the theorem breaks down into five lemmas. The first four lemmas show that if a formula is not provable then its negation has a nonstandard model. This part of the argument is rather straightforward and a reader who finds the conclusion convincing may prefer to skip it or skim over it. The last lemma shows how the nonstandard model produced by lemmas 1-4, (which satisfies the induction axioms) can be converted to a finite standard model.

We have left out the axioms and arguments pertaining to test programs of the A? type. However, the completeness theorem can be extended to include these programs if the axiom schema (12) $[A?]B \leftrightarrow (A \rightarrow B)$ is included.

Lemma 1:

1) If $\vdash A \rightarrow B$ then $\vdash [\alpha]A \rightarrow [\alpha]B$ and $\vdash <\alpha>A \rightarrow <\alpha>B$

2) $\vdash [\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$

3) $\vdash <\alpha>(A \vee B) \leftrightarrow <\alpha>A \vee <\alpha>B$

4) $\vdash [\alpha]A \wedge <\alpha>B \rightarrow <\alpha>(A \wedge B)$.

[Note that 1), 3) imply: if $\vdash C \leftrightarrow (A \vee B)$ then $\vdash <\alpha>C \leftrightarrow (<\alpha>A \vee <\alpha>B)$.]

Proof:

1) If $\vdash A \to B$ then $\vdash [\alpha](A \to B)$ by the rule gen. Hence $\vdash [\alpha]A \to [\alpha]B$ by axiom 2 and modus ponens. Also $\vdash \neg B \to \neg A$, hence $\vdash [\alpha]\neg B \to [\alpha]\neg A$, so $\vdash \langle \alpha \rangle A \to \langle \alpha \rangle B$.

2) Since $\vdash A \wedge B \to A$ we have $\vdash [\alpha](A \wedge B) \to [\alpha]A$ by 1) above. Similarly, $\vdash [\alpha](A \wedge B) \to [\alpha]B$. So $\vdash [\alpha](A \wedge B) \to [\alpha]A \wedge [\alpha]B$ using a tautology and mp. On the other hand we have $\vdash A \to (B \to (A \wedge B))$. Hence $\vdash [\alpha]A \to [\alpha](B \to A \wedge B)$ by 1) above. So $\vdash [\alpha]A \to ([\alpha]B \to [\alpha](A \wedge B))$ by axiom 2, tautologies and mp. Hence $\vdash [\alpha]A \wedge [\alpha]B \to [\alpha](A \wedge B)$. This yields 2.

3) This is just a variant of 2 above if we remember that $\langle \alpha \rangle$ is $\neg[\alpha]\neg$.

4) We have, by 3, $\vdash \langle \alpha \rangle B \leftrightarrow \langle \alpha \rangle (A \wedge B) \vee \langle \alpha \rangle (\neg A \wedge B)$, also $\vdash [\alpha]A \to \neg \langle \alpha \rangle(\neg A \wedge B)$. Hence $\vdash [\alpha]A \wedge \langle \alpha \rangle B \to \langle \alpha \rangle(A \wedge B)$

Lemma 2:

1) $\vdash [(\alpha \cup \beta)^{-1}]A \leftrightarrow [\alpha^{-1} \cup \beta^{-1}]A.$

2) $\vdash [(\alpha\beta)^{-1}]A \leftrightarrow [\beta^{-1}\alpha^{-1}]A$

3) $\vdash [(\alpha^{-1})^{-1}]A \leftrightarrow [\alpha]A$

Proof:

1) a) $\to$. By axiom 3 it is sufficient to prove that $[(\alpha \cup \beta)^{-1}]A$ is inconsistent with $\langle \alpha^{-1} \rangle \neg A$ as well as with $\langle \beta^{-1} \rangle \neg A$. By similarity we may consider just $\langle \alpha^{-1} \rangle \neg A$. From $\langle \alpha^{-1} \rangle \neg A$ we may infer (i) $\langle \alpha^{-1} \rangle (\neg A \wedge [\alpha \cup \beta] \langle (\alpha \cup \beta)^{-1} \rangle \neg A)$ since $\neg A \to [\alpha \cup \beta] \langle (\alpha \cup \beta)^{-1} \rangle \neg A$ is an axiom 8.

Also from $[(\alpha \cup \beta)^{-1}]A$ we may conclude, using axiom 9,

(ii) $[\alpha^{-1}]\langle \alpha \rangle [(\alpha \cup \beta)^{-1}]A$.

Combining (i), (ii) by Lemma 1, (4), we get

$\langle \alpha^{-1} \rangle [\neg A \wedge [\alpha \cup \beta] \langle (\alpha \cup \beta)^{-1} \rangle \neg A \wedge \langle \alpha \rangle [(\alpha \cup \beta)^{-1}]A]$.

Hence,
$< \alpha^{-1} > [\neg A \wedge [\alpha] < (\alpha \cup \beta)^{-1} > \neg A \wedge < \alpha > [(\alpha \cup \beta)^{-1}]A]$, and hence
$< \alpha^{-1} > < \alpha > (< (\alpha \cup \beta)^{-1} > \neg A \wedge [(\alpha \cup \beta)^{-1}]A)$ which can easily be disproved.

b) $\leftarrow$. It is sufficient to show that $[\alpha^{-1}]A$, $[\beta^{-1}]A$, $< (\alpha \cup \beta)^{-1} > \neg A$ are inconsistent.

From the first two hypotheses we may conclude
$[(\alpha \cup \beta)^{-1}] < \alpha \cup \beta > ([\alpha^{-1}]A \wedge [\beta^{-1}]A)$
and hence, by the third hypothesis,
$< (\alpha \cup \beta)^{-1} > (\neg A \wedge < \alpha \cup \beta > ([\alpha^{-1}]A \wedge [\beta^{-1}]A))$.
Hence,
$< (\alpha \cup \beta)^{-1} > (\neg A \wedge (< \alpha > ([\alpha^{-1}]A \wedge [\beta^{-1}]A) \vee < \beta > ([\alpha^{-1}]A \wedge [\beta^{-1}]A) \wedge$
$[\alpha] < \alpha^{-1} > \neg A \wedge [\beta] < \beta^{-1} > \neg A))$.
This yields
$< (\alpha \cup \beta)^{-1} > (< \alpha > ([\alpha^{-1}]A \wedge < \alpha^{-1} > \neg A) \vee < \beta > ([\beta^{-1}]A \wedge < \beta^{-1} > \neg A)$
which can also be easily disproved.

(2) a) $\rightarrow$. It is sufficient to prove that $[(\alpha;\beta)^{-1}]A$, $< \beta^{-1}\alpha^{-1} > \neg A$ are inconsistent. From $[(\alpha \beta)^{-1}]A$ we get, by an axiom, $[\beta^{-1}] < \beta > [(\alpha \beta)^{-1}]A$ and hence using the axiom $B \rightarrow [\alpha^{-1}] < \alpha > B$ where B is $< \beta > [(\alpha \beta)^{-1}]A$, we get, $[\beta^{-1}][\alpha^{-1}] < \alpha > < \beta > [(\alpha \beta)^{-1}]A$. Combining that with $< \beta^{-1} > < \alpha^{-1} > \neg A$ from the second hypothesis, we get
$< \beta^{-1} > < \alpha^{-1} > (\neg A \wedge < \alpha > < \beta > [(\alpha \beta)^{-1}]A)$.

Now we use the fact that $\neg A \rightarrow [\alpha \beta] < (\alpha \beta)^{-1} > \neg A$ is an axiom and get after some manipulations,
$< \beta^{-1} > < \alpha^{-1} > < \alpha > < \beta > ([(\alpha \beta)^{-1}]A \wedge < (\alpha \beta)^{-1} > \neg A))$ which can be disproved.

b) $\leftarrow$. It is sufficient to prove that $[\beta^{-1}\alpha^{-1}]A$ and $< (\alpha \beta)^{-1} > \neg A$ are inconsistent. I.e., $[\beta^{-1}][\alpha^{-1}]A$ and $< (\alpha \beta)^{-1} > \neg A$ are inconsistent.
From $[\beta^{-1}][\alpha^{-1}]A$ we can deduce

i) $[(\alpha\beta)^{-1}] < \alpha\ \beta > [\beta^{-1}][\alpha^{-1}]A$ using axiom 9.

On the other hand, we have as axioms

$\neg A \rightarrow [\alpha] < \alpha^{-1} > \neg A$ and $<\alpha^{-1}> \neg A \rightarrow [\beta] <\beta^{-1}> <\alpha^{-1}> \neg A$. By gen and tautologies and mp we get

$\neg A \rightarrow [\alpha][\beta] < \beta^{-1} >< \alpha^{-1} > \neg A$ and since we have $< (\alpha\ \beta)^{-1}) > \neg A$ we have also

ii) $< (\alpha\ \beta)^{-1} > [\alpha][\beta] < \beta^{-1} > < \alpha^{-1} > \neg\ A$. From (i), (ii) we can get

$< (\alpha\ \beta)^{-1} > < \alpha > < \beta > [[\beta^{-1}][\alpha^{-1}]A \wedge < \beta^{-1} > < \alpha^{-1} > \neg A]$ which can be disproved.

(3) Similar. We omit the details.

Cor: 1), 2), 3), above hold with $< >$ instead of $[\ ]$. E.g.

$\vdash <(\alpha \cup \beta)^{-1}> A \leftrightarrow <\alpha^{-1} \cup \beta^{-1}> A$ etc. This follows easily from the lemma when we substitute $\neg A$ for A and use contrapositives.

Definition 4: A pseudomodel $\mathfrak{M}$ consists of a finite tree of boxes with an origin box b_0 such that (a) each box has a finite number of formulas, but at least one, written in it, (b) some pairs of boxes are connected by an arrow marked with a program letter a or a^{-1} or with an α^* for some α, (c) every box is uniquely reachable from b_0 by following the arrows.

Definition 5: If $\mathfrak{M}_1$, $\mathfrak{M}_2$ are pseudomodels then $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ iff (1) every box of $\mathfrak{M}_1$ is a box of $\mathfrak{M}_2$, (2) if a box b contains a formula A in $\mathfrak{M}_1$ it also contains the same formula in $\mathfrak{M}_2$, (3) if b, b' are connected by an arrow in $\mathfrak{M}_1$, they are connected by the same arrow in $\mathfrak{M}_2$. (b_0 is, of course, the common origin box.)

Thus the only way $\mathfrak{M}_2$ can differ from $\mathfrak{M}_1$ is by having more formulas, or more boxes and arrows.

In Definition 6 below, a semi-atomic program is one which is either a program letter a or of the form a^{-1} or of the form α^* for any α.

Definition 6: A pseudomodel $\mathfrak{M}$ immediately reduces to $\mathfrak{M}_1$ (to $\mathfrak{M}_1, \mathfrak{M}_2$, in symbols, $\mathfrak{M} \prec \mathfrak{M}_1$ or $\mathfrak{M} \prec \mathfrak{M}_1, \mathfrak{M}_2$, respectively) iff at least one of the following holds.

a) Some box b of $\mathfrak{M}$ contains a formula $A \vee B$ and $\mathfrak{M}_1, \mathfrak{M}_2$ are just like $\mathfrak{M}$ except that box b in $\mathfrak{M}_1$ contains the additional formula A and in $\mathfrak{M}_2$, the additional formula B.

b) $\mathfrak{M}_1$ is obtained from $\mathfrak{M}$ by adding A to some box of $\mathfrak{M}$, where $\vdash A$.

c) $\mathfrak{M}_1$ is obtained from $\mathfrak{M}$ by adding the formula B to a box b of $\mathfrak{M}$ where b already contains A, $A \rightarrow B$.

d) $\mathfrak{M}$ contains $b \overset{\alpha}{\rightarrow} b'$ and $\mathfrak{M}_1$ is obtained from $\mathfrak{M}$, by adding to b' the formula A, where b already contains, in $\mathfrak{M}$, the formula $[\alpha]A$, and α is semi-atomic.

e) $\mathfrak{M}_1$ is obtained from $\mathfrak{M}$ by creating a new box b' and arrows $b \overset{\alpha}{\rightarrow} b'$ and writing A in b' where $\mathfrak{M}$ already contains box b and box b in $\mathfrak{M}$ contains the formula $<\alpha>A$ where α is <u>semiatomic</u>.

It is easily seen that if $\mathfrak{M}$ is a pseudomodel, so are $\mathfrak{M}_1$, $\mathfrak{M}_2$ (so is $\mathfrak{M}_1$).

<u>Definition 7</u>: If $\mathfrak{M}$ is a pseudomodel, then we define, for each box b, the <u>describing</u> formula A_b of b, and the describing formula $A_{\mathfrak{M}}$ of $\mathfrak{M}$ as follows:

1) If b has no arrows leading out from it then $A_b = \bigwedge A_i : A_i \in b$

2) If b has arrows $\alpha_1, \ldots, \alpha_k$ to boxes $b_1, \ldots, b_k$ and $A_{b_1}, \ldots, A_{b_k}$ are already defined, then

$$A_b = (\bigwedge A_i : A_i \in b) \wedge (\bigwedge_{i=1}^{k} <\alpha_i> A_{b_i})$$
$$A_{\mathfrak{M}} = A_{b_0}.$$

<u>Definition 8</u>: A pseudomodel $\mathfrak{M}$ is <u>inconsistent</u> if $\vdash \neg A_{\mathfrak{M}}$.

<u>Lemma 3</u>: 1) If $\mathfrak{M}$ reduces to $\mathfrak{M}_1$ then $\vdash A_{\mathfrak{M}} \leftrightarrow A_{\mathfrak{M}_1}$.

2) If $\mathfrak{M}$ reduces to $\mathfrak{M}_1, \mathfrak{M}_2$ then $\vdash A_{\mathfrak{M}} \leftrightarrow (A_{\mathfrak{M}_1} \vee A_{\mathfrak{M}_2})$.

<u>Proof</u>: The proof is straightforward from lemma 1. The pseudomodel $\mathfrak{M}$ is a tree with b_0 as root and a box b is a leaf (end node) if there are no arrows coming out of it. Define the height h(b) of a box b to be 0 if b is a leaf and to be $\max(h(b_i)+1)$ otherwise, where the max is taken over those boxes b_i which are recipients of arrows coming out of b. Then b_0 has the greatest height. Also say b is <u>above</u> c, if the unique path from b_0 to c passes through b. (b may be c or b_0.)

1) Now if $\mathfrak{M}$ reduces to $\mathfrak{M}_1$ then $\mathfrak{M}_1$ was obtained from $\mathfrak{M}$ by performing one of the actions (b)-(e) of definition 6. Consider action (c). If A_b, A_b^1 are the describing formulae of b in $\mathfrak{M}$, $\mathfrak{M}_1$, respectively, clearly $\vdash A_b \leftrightarrow A_b^1$ and hence by induction on height, and lemma 1, part 1, for all boxes c above b, $\vdash A_c \leftrightarrow A_c^1$. Hence $\vdash A_{b_0} \leftrightarrow A_{b_0}^1$, i.e., $\vdash A_{\mathfrak{M}} \leftrightarrow A_{\mathfrak{M}_1}$.

The other cases are similar.

2) Suppose action (a) of definition 6 was applied to a formula $A \vee B$ in box b of $\mathfrak{M}$ to get $\mathfrak{M}_1$, $\mathfrak{M}_2$. Then clearly, for boxes c strictly below b, A_c, A_c^1, A_c^2 are identical. At b itself we have $\vdash A_b \leftrightarrow (A_b^1 \vee A_b^2)$ and for all c, above b, the same will hold, by induction on height. Hence with $c = b_0$ we get $\vdash A_{\mathfrak{M}} \leftrightarrow (A_{\mathfrak{M}_1} \vee A_{\mathfrak{M}_2})$.

<u>Cor</u>: 1) If $\mathfrak{M}$ reduces to $\mathfrak{M}_1$ and $\vdash \neg A_{\mathfrak{M}_1}$ then $\vdash \neg A_{\mathfrak{M}}$.

2) If $\mathfrak{M}$ reduces to $\mathfrak{M}_1, \mathfrak{M}_2$ and $\vdash \neg A_{\mathfrak{M}_1}$, $\vdash \neg A_{\mathfrak{M}_2}$ then $\vdash \neg A_{\mathfrak{M}}$.

<u>Remark</u>: The pseudo-models that we defined in definition 3 are trees of boxes and arrows. However, the proof search procedure described in the next lemma generates a tree of pseudo-models, i.e., in which each pseudo-model is a node. König's lemma is applied to this "supertree."

<u>Lemma 4</u>: Given a formula C, either $\vdash \neg C$ or there is a nonstandard model of C which satisfies the induction axioms. (Axioms 10, def. 3)

<u>Proof</u>: Definition 6 can be regarded as describing five different kinds of actions that may be performed on a pseudomodel $\mathfrak{M}$ to yield one or two expanded pseudomodels. (Think of the new pseudomodels as <u>replacing</u> the old one.)

We start with

$$\mathfrak{M}_1^1 = \boxed{C}\ b_0$$

and then we carry out the actions a-e of definition 7 in some systematic way so that every possible action is eventually performed. If at some stage some pseudomodel is inconsistent, no further work is done on it, but it remains. Then we have, by König's lemma, two possibilities:

a) there is a stage when all existing pseudomodels are inconsistent. Suppose $\mathfrak{M}^n_1, \ldots, \mathfrak{M}^n_k$ are the pseudomodels and $A^n_1, \ldots, A^n_k$ are their describing formulae, then by def 8, $A^n_1, \ldots,$ A^n_k and by cor. to lemma 3, C.

b) there is no such stage, in which case, by König's lemma, we have an increasing sequence $\mathfrak{M}^i$ of pseudomodels in which every possible action is eventually performed. Let $\mathfrak{M}_0$ be the limit of these $\mathfrak{M}^i$ in the sense that the boxes and arrows of $\mathfrak{M}_0$ include those of each $\mathfrak{M}^i$ and a box b in $\mathfrak{M}_0$ contains a formula A iff it contains that formula in some $\mathfrak{M}^i$. $\mathfrak{M}_0$ will have, in general, countably many boxes and each box will have countably many formulas written on it.

We observe the following facts:

a) If $\vdash A$ then A is written on every box b. (by action b of definition 6).

b) the set of formulas written on b is closed under modus ponens (by action c).

c) for each B either B or $\neg B$ is written on b (since $B \vee \neg B$ will be written and eventually action a will be performed on $B \vee \neg B$).

d) if boxes b, b' are connected by a (semi-atomic) arrow α and $[\alpha]B$ is written on b, then B is written on b'. (By action d.)

e) if $< \alpha > B$ is written on b <u>and</u> α is semi-atomic, then for some b', $b \overset{\alpha}{\to} b'$ and B is written on b' (by action e).

f) The set of formulas written at any box is consistent i.e if $A_1, \ldots, A_n$ are written at b then $\nvdash \neg(A_1 \wedge \ldots \wedge A_n)$. Otherwise our chain of pseudo models would have been "killed."

Now we define a nonstandard model $\mathfrak{M}_1$ as follows. Let W = the set of boxes of $\mathfrak{M}_0$. For each α which is a program letter a, we define

$R_\alpha = \{(b,b') \mid$ there is an a-arrow from b to b'$\} \cup$

$\{(b',b) \mid$ there is an a^{-1}-arrow from b to b'$\}$

then $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$, $R_{\alpha^{-1}} = (R_\alpha)^{-1}$ and $R_{\alpha;\beta} = R_\alpha \cdot R_\beta$. If α is of the form β^*, then let $S_\alpha = \{(b,b') \mid$ there is an α arrow from b to b'$\}$

and

R_α = reflexive transitive closure of $S_\alpha \cup R_\beta \cup (S_\gamma)^{-1}$ where γ is $(\beta^{-1})^*$

Finally, v(b,P) = T iff P is written on box b.

It is straightforward to check that the $\mathfrak{M}_1$ obtained this way is a nonstandard model of C:

Claim 1: If $(b,b') \in R_\alpha$ and $[\alpha]A$ is written at b in $\mathfrak{M}_0$ then A is written at b' in $\mathfrak{M}_0$.

Proof of Claim: Define the height of α as follows:

$h(\alpha) = 0$ if α is atomic.
$h(\alpha \cup \beta) = h(\alpha;\beta) = \max(h(\alpha), h(\beta)) + 1$.
$h(\alpha^{-1}) = h(\alpha^*) = h(\alpha) + 1$.

We shall prove the claim by induction on the height of α.

1) $h(\alpha) = 0$, i.e., α is atomic. Then α is also semi-atomic. Either there is a α-arrow from b to b' and A is written at b' by action (d). Or else there is an α^{-1} arrow from b' to b. If A were not written at b' then $\neg A$ would be and hence $[\alpha^{-1}]\langle\alpha\rangle\neg A$ would be, by axiom 9. But this and action d would yield $\langle\alpha\rangle\neg A$ at b .

2) $h(\alpha) > 1$.
(i) $\alpha = (\beta \cup \gamma)$. Then by actions (b), (c), $[\beta]A \wedge [\gamma]A$ and hence $[\beta]A$ and $[\gamma]A$ are written at b. Also $(b,b') \in R_\beta$ or $(b,b') \in R_\gamma$. So by induction hypothesis, A is written at b'.

(ii) $\alpha = (\beta;\gamma)$. By definition 6, actions (b), (c), $[\beta][\gamma]A$ is written at b. Moreover, there is a box b'' such that $(b,b'') \in R_\beta$ and $(b'',b') \in R_\gamma$. By induction hypothesis, $[\gamma]A$ is written at b'' and so A at b'.

(iii) $\alpha = \beta^{-1}$. Since $(b,b') \in R_\alpha$, $(b',b) \in R_\beta$. If A were not written at b' then $\neg A$ would be and hence $[\beta]\langle\beta^{-1}\rangle\neg A$ would be written . By induction hypothesis, since $h(\beta) < h(\alpha)$, $\langle\beta^{-1}\rangle\neg A$ would be written at b contradicting observation f. Hence A must be written at b'.

(iv) $\alpha = \beta^*$. Since $(b,b') \in R_{\beta^*}$, there is a chain $b_1,\ldots,b_n$ such that $b = b_1$, $b' = b_n$ a nd for all $i < n$, either $(b_i,b_{i+1}) \in R_\beta$ or $(b_i,b_{i+1}) \in S_\alpha$ or $(b_{i+1},b_i) \in S_{(\beta^{-1})^*}$. Then we can show by induction on i that for all $i \le n$, b_i has $[\beta^*]A$ written on it. This is true for $b = b_1$. Suppose true for b_i. Now if (b_i,b_{i+1}) is in R_β then since b_i also has $[\beta][\beta^*]A$, by axioms 7 and 5, b_{i+1} has $[\beta^*]A$ by induction hypothesis, using $h(\beta) < h(\beta^*)$. If (b_i,b_{i+1}) is in S_α then we notice that b_i also has $[\beta^*][\beta^*]A$ by axiom 7 and hence b_{i+1} has $[\beta^*]A$ by action d, def. 6. Finally if $(b_{i+1},b_i) \in S_\gamma$ where $\gamma = (\beta^{-1})^*$, then b_{i+1} must have $[\beta^*]A$, otherwise it has $\neg[\beta^*]A$ and hence also $[(\beta^*)^{-1}]\langle\beta^*\rangle\neg[\beta^*]A$ at b_{i+1}, by axiom 11, and hence $\langle\beta^*\rangle\neg[\beta^*]A$ (by action d, def. 6) at b_i. But b_i already contains $[\beta^*][\beta^*]A$, contradiction. So b_{i+1} must have $[\beta^*]A$ in any case. Thus b_n has $[\beta^*]A$, and hence A (axiom 5)

Claim 2: for each box b and formula A, A is written on b in $\mathfrak{M}_0$ iff $\mathfrak{M}_1$, b $\models$ A, (where α^* is treated as atomic). In particular, all axioms hold at all b in $\mathfrak{M}_1$.

Proof by induction on the complexity of A: (IH = induction hypothesis)

1) If A is atomic then this holds trivially.

2) If $A = \neg B$ then we have $\mathfrak{M}_1, b \models A$ iff $\mathfrak{M}_1, b \not\models B$ iff (IH) B is not written on b iff $\neg B$ is written on b.

3) If $A = B \vee C$, $\mathfrak{M}_1, b \models A$ iff $\mathfrak{M}_1, b \models B$ or $\mathfrak{M}_1, b \models C$ iff B or C is written on b (IH) iff $B \vee C$ is written on b. (Otherwise $\neg(B \vee C)$ would be written)

4) $A = [\alpha]\beta$. If A is written at b, then for all b' such that $(b,b') \in R_\alpha$, B is written at b'. (Claim 1). Hence by induction hypothesis, B holds at all such boxes and hence A holds at b.

So suppose now that A is not written at b. Then $\neg[\alpha]B$ and $<\alpha>\neg B$ are written at b. Now we get several cases depending on $h(\alpha)$ defined before.

(a) $h(\alpha) = 0$. This is immediate by action (e) of definition 6.

(b) $h(\alpha) > 0$.

(i) $\alpha = \beta^*$. This is immediate also since β^* is semi-atomic by action (e).

(ii) $\alpha = \beta \cup \gamma$. Since $\neg[\alpha]B$ is written at b, so is $\neg([\beta]B \wedge [\gamma]B)$ and hence $<\beta>\neg B \vee <\gamma>\neg B$. Hence one of the latter is written and holds by induction hypothesis. Hence $<\beta>\neg B \vee <\gamma>\neg B$ holds and hence $<\alpha>\neg B$ holds so $[\alpha]B$ does not.

(iii) $\alpha = \beta^{-1}$. If β is a program letter, there is a β^{-1} arrow to a b' where $\neg B$ is written so $(b,b') \in R_\alpha$ and $[\alpha]B$ doesn't hold at b. If β is not a program letter, β is $\gamma \cup \delta$ or $\gamma;\delta$ or γ^*. In the first two cases, respectively, $<\gamma^{-1}>\neg B \vee <\delta^{-1}>\neg B$ and $<\delta^{-1}><\gamma^{-1}>\neg B$ are written at b also. Thus we are reduced to the simpler γ^{-1}, δ^{-1} cases and can apply induction hypothesis. In the last case where α is $(\gamma^*)^{-1}$, $<(\gamma^{-1})^*>\neg B$ is also written at b and we are reduced to the semi-atomic case (1).

(iv) $\alpha = (\beta;\gamma)$. This is similar. We omit the details.

This completes the proof of claim 2 and lemma 4.

<u>Lemma 5</u>: If a formula C has a nonstandard model $\mathfrak{M}_1$ which satisfies the induction axioms 10, then it has a finite standard model.

<u>Proof</u>: We define the notion of "closed set" as in the Fischer-Ladner paper, p. 288, except for remarking that we treat $[\alpha]$ rather than $<\alpha>$ as the basic operation and that we add

7) $[(\alpha;\beta)^{-1}]A \in S \to [\beta^{-1}][\alpha^{-1}]A \in S$

8) $[(\alpha \cup \beta)^{-1}]A \in S \to [\alpha^{-1}]A,\ [\beta^{-1}]A \in S$

9) $[(\alpha^*)^{-1}]A \in S \to [(\alpha^{-1})^*]A \in S.$

We can show that given the formula C, there is a finite S containing C which is closed in our sense.

We now define the factor model $\overline{\mathfrak{M}}$ in the same way. Let

$b \equiv b'$ iff for all $A \in S$, $\mathfrak{M}_1, b \models A$ iff $\mathfrak{M}_1, b' \models A$

$\overline{W} = \{\overline{b} \mid b \in W\}$ and for α a program letter or of the form β^*,

$\overline{R}_\alpha = \{(\overline{b}, \overline{b'}) \mid (b,b') \in R_\alpha\}$ where $\overline{b}, \overline{b'}$ are the equivalence classes of b,b' under $\equiv$. Also let $\overline{R}_{\alpha;\beta} = \overline{R}_\alpha \bullet \overline{R}_\beta$, $\overline{R}_{\alpha\cup\beta} = \overline{R}_\alpha \cup \overline{R}_\beta$ and $\overline{R}_{\alpha^{-1}} = (\overline{R}_\alpha)^{-1}$.

Note now that for each α, $\overline{R}_{\alpha^*}$ is reflexive and contains $\overline{R}_\alpha$, but need not be transitive. Just as in the Fischer-Ladner paper, $\overline{\mathfrak{M}}$ is a model of C <u>provided</u> we don't insist that $(\overline{R}_\alpha)^* = \overline{R}_{\alpha^*}$. Note also that for α, non-semi-atomic, $(x,y) \in R_\alpha$ implies $(\overline{x},\overline{y}) \in \overline{R}_\alpha$ but not necessarily the other way round.

<u>Claim 1</u>: $\overline{R}_{\alpha^*} \subseteq (\overline{R}_\alpha)^*$

<u>Proof of Claim</u>: For each point $\overline{x}$ of $\overline{\mathfrak{M}}$, let

$T_{\overline{x}} = (\bigwedge A : A \in S \wedge \mathfrak{M}_1, x \models A) \wedge (\bigwedge \neg A : A \in S \wedge \mathfrak{M}_1, x \models \neg A).$

This notation is not ambiguous since $\overline{x} = \overline{y}$ iff $T_{\overline{x}} = T_{\overline{y}}$ so $T_{\overline{x}}$ depends only on $\overline{x}$. Indeed we may identify $T_{\overline{x}}$ with $\overline{x}$.

Suppose now that there is a $(T_{\overline{x}}, T_{\overline{y}})$ in $\overline{R}_{\alpha^*}$ not in $(\overline{R}_\alpha)^*$. Then let $T = \bigvee T_{\overline{z}} : (T_{\overline{x}}, T_{\overline{z}}) \in (\overline{R}_\alpha)^*$. We can assume without loss of generality that $(x,y) \in R_{\alpha^*}$ was the reason why $(T_{\overline{x}}, T_{\overline{y}}) \in \overline{R}_{\alpha^*}$. Now we have in $\mathfrak{M}_1$ $\mathfrak{M}_1, x \not\models [\alpha^*]T$ since $\mathfrak{M}_1, x \models \langle\alpha^*\rangle T_{\overline{y}}$ and $T_{\overline{y}}$ was not provided for in T. (I.e., $\vdash T_{\overline{y}} \to \neg T$.) But we do have

$\mathfrak{M}_1, x \models T \wedge [\alpha^*](T \rightarrow [\alpha]T) \rightarrow [\alpha^*]T$ and we also have $\mathfrak{M}_1, x \models T$.

Hence $\mathfrak{M}_1, x \models \neg[\alpha^*](T \rightarrow [\alpha]T)$ and so

$\mathfrak{M}_1, x \models <\alpha^*> (T \wedge <\alpha> \neg T)$.

Let w,z be such that $(x,z) \in R_{\alpha^*}$ and $\mathfrak{M}_1, z \models T \wedge <\alpha> \neg T$ and $(z,w) \in R_\alpha$ and $\mathfrak{M}_1, w \models \neg T$.

Then $(T_{\bar{x}}, T_{\bar{z}}) \in (\bar{R}_\alpha)^*$ and $(T_{\bar{z}}, T_{\bar{w}}) \in \bar{R}_\alpha$ but then $(T_{\bar{x}}, T_{\bar{w}}) \in (\bar{R}_\alpha)^*$ which is a contradiction since $T_{\bar{w}}$ is incompatible with T.

<u>Claim 2</u>: If $(T_{\bar{x}}, T_{\bar{z}}) \in (\bar{R}_\alpha)^* - \bar{R}_{\alpha^*}$, and $\bar{R}_{\alpha^*}$ is increased to include $(T_{\bar{x}}, T_{\bar{z}})$ then this does not change the semantics in $\bar{\mathfrak{M}}$ as far as formulas of S are concerned.

<u>Proof of Claim</u>: By induction on the complexity of formulae, we can show that if any formula of S changes its truth value then the simplest such formula must be of the form $<\alpha^*>$ A where A changes its truth value nowhere and $<\alpha^*>$ A becomes true at $\bar{x}$ whereas it was false before.

Now if $<\alpha^*>$ A was not true before at $\bar{x}$ then $[\alpha^*]\neg A$ was true. Let $T_1 = T_{\bar{x}}, T_2, \ldots, T_n = T_{\bar{z}}$ be such that $(T_i, T_{i+1}) \in \bar{R}_\alpha$ for all $i < n$.

Then since $[\alpha^*]\neg A \in T_1$, so $[\alpha][\alpha^*]\neg A \in T_1$ and hence $[\alpha^*]\neg A \in T_2$, etc. Continuing this way we get $[\alpha^*]\neg A \in T_n$ so $\neg A \in T_n$, i.e., $\neg A \in T_{\bar{z}}$, and the value of A changed nowhere. Thus $\neg A$ is still true at $T_{\bar{z}}$ and cannot make $<\alpha^*>$ A true at $T_{\bar{x}}$.

Starting with the simplest α^*, we continue to augment the various $\bar{R}_{\alpha^*}$ until $\bar{R}_{\alpha^*}$ becomes equal to $(\bar{R}_\alpha)^*$ ($\bar{\mathfrak{M}}$ is finite). Then we proceed to the next simplest α^* and so on. Ultimately, for all the α^* involved in S we have $\bar{R}_{\alpha^*} = (\bar{R}_\alpha)^*$ and we have a standard, finite, model for our formula C.

This completes the proof of the completeness theorem (theorem 1).

§ 2. In this section we show that the axiom schema (11) $[(\alpha^*)^{-1}]A \leftrightarrow [(\alpha^{-1})^*]A$ is eliminable. This proof is due to Vaughan Pratt.

Lemma 6: The following six are derived rules of the formal system of Definition 3, less axioms 11.

1) $\dfrac{A \to [\alpha]B}{<\alpha^{-1}> A \to B}$ 2) $\dfrac{A \to [\alpha^{-1}]B}{<\alpha> A \to B}$

3) $\dfrac{<\alpha> A \to B}{A \to [\alpha^{-1}]B}$ 4) $\dfrac{<\alpha^{-1}> A \to B}{A \to [\alpha]B}$

5) $\dfrac{A \to [\alpha]A}{A \to [\alpha^*]A}$ 6) $\dfrac{<\alpha> A \to A}{<\alpha^*> A \to A}$.

1) We have $\vdash \neg B \to [\alpha^{-1}] <\alpha> \neg B$ (axiom)
and hence $\vdash <\alpha^{-1}> [\alpha]B \to B$ (contraposition).

Also

$\vdash <\alpha^{-1}> A \to <\alpha^{-1}> [\alpha]B$ from the hypothesis and lemma 1.1.

Hence

$\vdash <\alpha^{-1}> A \to B$.

2) is similar to (1). Just interchange α^{-1} and α.

3) We have $\vdash [\alpha^{-1}] <\alpha> A \to [\alpha^{-1}]B$ from the hypothesis by Lemma 1.1.

Also

$\vdash A \to [\alpha^{-1}] <\alpha> A$ (axiom).

Hence

$\vdash A \to [\alpha^{-1}]B$.

4) is similar to (3). Interchange α, α^{-1}.

5) We have from the hypothesis by generalization

$\vdash [\alpha^*](A \to [\alpha]A)$.

Combining this with a suitable induction axiom, we get,

$\vdash A \to [\alpha^*]A$.

6) is just 5) with $\neg A$ replacing A and using contraposition.

Note that the rules apply only to purely logical theorems. We do <u>not</u> have the axiom schema

$$(A \to [\alpha]B) \to (<\alpha^{-1}>A \to B)$$

for instance.

<u>Lemma 7</u>: 1) $\vdash[(\alpha^{-1})^{*}]A \to [(\alpha^{*})^{-1}]A$

2) $\vdash[(\alpha^{*})^{-1}]A \to [(\alpha^{-1})^{*}]A.$

<u>Proof</u>: 1) $\vdash[(\alpha^{-1})^{*}]A \to [\alpha^{-1}][(\alpha^{-1})^{*}]A$ by axioms.

Hence $\vdash <\alpha> [(\alpha^{-1})^{*}]A \to [(\alpha^{-1})^{*}]A$ by lemma 6.

Hence $\vdash <\alpha^{*}> [(\alpha^{-1})^{*}]A \to [(\alpha^{-1})^{*}]A.$

But $\vdash [(\alpha^{-1})^{*}]A \to A$ (axiom 6)

So $\vdash <\alpha^{*}> [(\alpha^{-1})^{*}]A \to A$

and $\vdash [(\alpha^{-1})^{*}]A \to [(\alpha^{*})^{-1}]A$ (lemma 6)

2) $\vdash\neg A \to [\alpha^{*}] < (\alpha^{*})^{-1}> \neg A$ (axiom)

Hence $\vdash <\alpha^{*}> [(\alpha^{*})^{-1}]A \to A$ (contraposition)

But $\vdash <\alpha^{*}> <\alpha> B \to <\alpha^{*}> B$

[from $\vdash[\alpha^{*}]\neg B \to [\alpha^{*}][\alpha]\neg B$].

Hence with $B = [(\alpha^{*})^{-1}]A$, we get

$\vdash <\alpha^{*}> <\alpha> [(\alpha^{*})^{-1}]A \to A$

and

$\vdash [(\alpha^{*})^{-1}]A \to [\alpha^{-1}][(\alpha^{*})^{-1}]A$ by lemma 6.

So ($\in$) $\vdash [(\alpha^{*})^{-1}]A \to [(\alpha^{-1})^{*}][(\alpha^{*})^{-1}]A$ by the same lemma.

However, we have $\vdash A \to <\alpha^{*}>A$ and hence by lemma 6,

$\vdash[(\alpha^{*})^{-1}]A \to A.$

Thus

$\vdash [(\alpha^{-1})^*][(\alpha^*)^{-1}]A \rightarrow [(\alpha^{-1})^*]A.$

Combining this with ($\in$) above, we get

$\vdash [(\alpha^*)^{-1}]A \rightarrow [(\alpha^{-1})^*]A.$

§3. PDL stands for "propositional dynamic logic." It is a rather general kind of modal logic and we know from Fischer-Ladner that the set of valid formulas is decidable. Nonetheless, the following result shows that in some ways, PDL is closer to first order logic.

<u>Definition 10</u>: A PDL structure $\mathfrak{M}$ is <u>recursive</u> if there is a finite alphabet Σ such that W is a recursive subset of Σ^*, for each program letter a the relation $R_a^{\mathfrak{M}}$ is recursive and for each propositional letter P, the set $U_P^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models P\}$ is recursive.

<u>Theorem 3</u>:

1) If $\mathfrak{M}$ is recursive then for each formula A the set $U_A^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models A\}$ is arithmetical and for each α, $R_\alpha^{\mathfrak{M}}$ is arithmetical.

2) There is a fixed recursive $\mathfrak{M}$ such that for each arithmetical subset $X \subseteq N$, there is a formula A of $\mathfrak{M}$ such that X is 1-1 reducible to U_A.

<u>Proof</u>:

1) We show by induction on the complexity of A,α that U_A, R_α are arithmetical. The following observations suffice.

a) $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$

b) $(x,y) \in R_{\alpha;\beta}$ iff $(\exists z)((x,z) \in R_\alpha \wedge (z,y) \in R_\beta)$

c) $(x,y) \in R_{\alpha^{-1}}$ iff $(y,x) \in R_\alpha$

d) $(x,y) \in R_{\alpha^*}$ iff $(\exists n)(\exists z)$ (z is a sequence number of length n $\wedge$

$$((z)_1)_1 = x \wedge ((z)_n)_2 = y \wedge$$

$$(\forall i \leq n)((((z)_i)_1, ((z)_i)_2 \in R_\alpha)$$

e) $\bigcup_{A \wedge B} = \bigcup_A \cap \bigcup_B$

f) $\bigcup_{\neg A} = W - \bigcup_A$

g) $\bigcup_{[\alpha]A} = \{w \mid (\forall w')((w,w') \in R_\alpha \rightarrow w' \in \bigcup_A)\}$.

In each case the left hand side is clearly arithmetical provided only that $R_\alpha, R_\beta, \bigcup_A, \bigcup_B$ are

2) We define $\mathfrak{M}$ as follows:

Let W = the set of all finite sequences of natural numbers of length ≥ 2. (We can clearly realize W as a recursive subset of $(\{a,b\})^*$).

Let $R_a = \{((x_1,\ldots,x_n), (x_1,\ldots, x_n, m)) \mid n \geq 2\}$.

Let $\bigcup_P = \{(x_1,\ldots,x_n) \mid n \geq 3 \wedge T_{n-3}(x_1,\ldots,x_n)\}$,

where $T_{\mathfrak{M}}$ is the appropriate Kleene T-predicate.

Given an arithmetical set X there exist z and k such that X is 1-1 reducible to

$Y = \{x \mid (\exists x_1)(\forall x_2)\ldots(\forall x_k)(\exists y)T_k(z,x,x_1\ldots x_k,y)\}$.

Let $A = \langle a \rangle [a] \langle a \rangle \ldots [a] \langle a \rangle P$

(where there are k+1 modalities in A).

Let f be the recursive function which reduces X to Y. Then

$x \in X$ iff $f(x) \in Y$ iff $(z,f(x)) \in \bigcup_A$.

Thus X is 1-1 reducible to $\bigcup_A$. Q.E.D.

Note that we have made no use of truth functional operations or operations on quantifiers in part 2 of the theorem. If these were used, no doubt the complexity of $\mathfrak{M}$ could be reduced.

Footnotes

1) Krister Segerberg [6] has announced a completeness result for the system without the inverse operation. He has indicated to us, however, that there is a gap in his proof (as of January 4, 1978) which he hopes to fill. Vaughan Pratt [5] has announced a Gentzen type completeness result.

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References

[1] M.J. Fischer and R.E. Ladner, "Propositional Modal Logic of Programs," Proceedings 9th Annual ACM Symposium on Theory of Computing, Boulder, Colorado, May 1977, 286-294.

[2] D. Harel, A.R. Meyer and V.R. Pratt, "Computability and Completeness in Logics of Programs," Ibid., 261-268.

[3] R. Parikh, "A Completeness Result for PDL," manuscript dated 11/29/77.

[4] V.R. Pratt, "Semantical Considerations in Floyd-Hoare Logic," Proceedings 17th IEEE Symposium on Foundations of Computer Science, 1976.

[5] V.R. Pratt, "A Practical Decision Method for Propositional Dynamic Logic," Proceedings 10th ACM Symposium on Theory of Computing, 1978, 326-337.

[6] K. Segerberg, "A Completeness Theorem in the Modal Logic of Programs," abstract, Notices of the American Mathematical Society, October, 1977, p. A552.