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RECURSION THEORETIC OPERATORS AND MORPHISMS ON NUMBERED SETS

Henk Barendregt
Giuseppe Longo

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by

Henk Barendregt[1]

and

Giuseppe Longo[2,3]

Dedicated to Buffee Lys Nelson on her Twelfth Birthday.

Abstract:

An operator is a map $\Phi: P\omega \to P\omega$. By embedding $P\omega$ in two natural ways into the λ-calculus model $P\omega^2$ (and T^ω) the computable maps on this latter structure induce several classes of recursion theoretic operators.

[1] Mathematical Institute, Budapestlaan 6, 3508 TA Utrecht, The Netherlands

[2] I.S.I., Corso Italia 40, 56100 Pisa, Italy

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§0. Introduction

With the notion of (pre complete) numbered set Ershov [3] gave a general framework for certain results in classical recursion theory. In his theory the notion of morphism is central. In [6] there is a definition of enumeration operators and (implicitly) of Turing operators. Although enumeration operators (restricted to the r.e. sets as numbered set) are morphisms, Turing operators are not even partial morphisms.

There is a natural correspondence between these (and other) classes of recursion theoretic operators and morphisms on an appropriate numbered set, via the constructive part of the λ-calculus models $P\omega^2$ and T^ω. The different classes of operators on $P\omega$ are effective continuous maps obtained by embedding $P\omega$ into $P\omega^2$ or T^ω in two natural ways, giving $P\omega$ either the Cantor or the Scott topology.

In particular Turing operators work on $P\omega$ with the Cantor topology. This is implicit in Nerode's theorem, see [6], p. 154, relating tt-reducibility to total Turing operators. Also a different proof will be given of a theorem in [6], p. 151, relating enumeration and Turing reducibility. Finally an interpolation result, in the sense of Algebra, will be proved for total Turing operators.

§1. The Models $P\omega, P\omega^2$ and T^ω

Let ω be the set of natural numbers with $P\omega$ as power set. $(P\omega, \subseteq)$ is a complete partial order (cpo) and so is $(P\omega^2, \sqsubseteq)$ with $< A,B > \sqsubseteq < A',B' >$ iff $A \subseteq A'$ $B \subseteq B'$; (these structures are even complete lattices). Cpo's X are always considered with the Scott topology, see [2], § 1 or [1], § 1.2. $[X \to X]$ is the cpo of continuous maps on X with the pointwise partial ordering. There is a binary operation on $P\omega$ such that $(P\omega, \cdot)$ is a **continuous λ-model,** i.e., a model of the λ-calculus in which exactly the continuous functions are representable, see [1], § 1.2.

Similarly one can make $P\omega^2$ into a continuous λ-model.

1.1 Notation $A, B, \ldots$ range over $P\omega; \overline{A} = \omega - A; a, b, \ldots$ range over $P\omega^2$; if $a =< A,B >$, then $a_- = A$ and $a_+ = B$; $n, m, .., i, j, .., p, q, \ldots$ range over ω; (n, m) is an effective bijective coding of ω^2 on ω; e_n is an effective enumeration of the finite elements of $P\omega^2$(i.e. of $\{a | a_-, a_+$ are finite $\}$), with $e_\circ =< \phi, \phi >$.

1.2 Proposition For $a, b \in P\omega^2$ define

$$a \cdot b =< \{m | \exists e_n \sqsubseteq b \ (n, m) \in a_-\}, \{m | \exists e_n \sqsubseteq b \ (n, m) \in a_+\} >$$

For $f \in [P\omega^2 \to P\omega^2]$ define

$$graph(f) =< \{(n, m) | m \in f(e_n)_-\}, \{(n, m) | m \in f(e_n)_+\} > .$$

Then $\cdot : P\omega^4 \to P\omega^2$ and graph $:[P\omega^2 \to P\omega^2] \to P\omega^2$ are continuous and moreover

$$graph(f) \cdot a = f(a).$$

In particular $(P\omega^2, \cdot)$ is a continuous λ-model.

Proof As for $P\omega$. •

In § 3 another continuous λ-model will be used, namely Plotkin's T^ω. One has

$$T^\omega = \{< A, B > | A \cap B = \phi\} \subseteq P\omega^2;$$

see [2] for the definition of application $(\cdot)$ and abstraction (graph) in this structure. These definitions use an effective enumeration $b_0, b_1, \ldots$ of the finite elements of T^ω.

1.3 Definition Let X be $P\omega, P\omega^2$ or T^ω.

(i) The computable part of X, notation X_c, is defined as follows:

$$P\omega_c = \{A | A \text{ is r.e. }\};$$

$$P\omega^2_c = (P\omega_c)^2;$$

$$T^\omega_c = T^\omega \cap P\omega^2_c.$$

Let $P\omega^2_c = \{\omega_i\}_{i \in \omega}$.

(ii) A map $f : X \to X$ is **computable** iff $\exists a \in X_c \, \forall x \in X \; f(x) = a \cdot x$.

1.4 Lemma Let X be as above and $f : X_c \to X$ be continuous. Then f has a unique continuous extension $\bar{f} : X \to X$.

Proof Define $\bar{f}(x) = \sqcup \{f(y) \mid y \sqsubseteq x, y \text{ finite}\}$. This is defined because the supremum is over a directed set. $\bar{f}$ is clearly the unique continuous extension of f. •

1.5 Definition A continuous $f : X_c \to X_c$ is called **computable** if its unique continuous extension $\bar{f} : X \to X$ is computable.

The following notions are due to Ershov.

1.6 Definition

(i) A **numbered set** is a structure (X, γ) where $\gamma : \omega \to X$ is a surjective map.

(ii) If (X, γ) and (X', γ') are numbered sets then $\mu : X \to X'$ is a partial **morphism** iff for some partial recursive $\psi : \omega \to \omega$ one has

$$\forall n\ \mu(\gamma(n)) \simeq \gamma'(\psi(n)).$$

(iii) If (X, γ) is a numbered set, then the Ershov topology on X has as base the collection

$$\{\gamma^{-1}(A) \mid A \text{ r.e.}\}.$$

For the definition of complete numbered set and special elements, see [3] or [8]. $P\omega_c$ with the standard enumeration $\gamma(n) = W_n$ forms a complete numbered set with special elements ϕ. Similarly $P\omega_c^2, T_c^\omega$ can be numbered to become complete numbered sets with special element $< \phi, \phi >$.

Morphisms between numbered sets are clearly continuous with respect to the Ershov topology. On our three numbered sets X_c, the morphisms coincide with the computable maps.

1.7 Generalized Rice-Shaphiro Theorem Let X be $P\omega, P\omega^2$ or T^ω. Then on X_c the Ershov topology coincides with the (trace of the) Scott topology.

Proof See [4], 2.5, where the result is proved in a more general context. •

1.8 Generalized Myhill-Sheperdron Theorem Let X be as above and $f : X_c \to X_c$. Then f is a morphism iff f is computable.

Proof ($\Rightarrow$) By 1.7 f is Scott continuous. An easy computation shows that $graph(f) \in X_c$.

($\Leftarrow$) Let $f(a) = b \cdot a$ with $b \in X_c$. Then f is a morphism, since an index of $b \cdot a$ can be computed uniformly from one of a. •

The following lemma is needed in § 3.

1.9 Lemma Any computable $f : T^\omega \to T^\omega$ can be extended to a computable $f^\sim : P\omega^2 \to P\omega^2$.

Proof Let $b = \lambda x \cdot f(x)$; then $b \in T_c^\omega$. Let h be the recursive function such that $e_{h(n)} = b_n$. Define

$$b^\sim = < \{(h(n), m) | (-n; m) \in b\_\}, \{(h(n), m) | (+n; m) \in b\_\} >, f^\sim(a) = b^\sim \cdot a \text{ in } P\omega^2.$$

See [2], §1 for notation. An easy computation shows that $f^\sim | T^\omega = f$, use [2], Lemma 1.6. •

§2. The $\Delta\,\bullet$-Operators

In order to define the recursion theoretic operators on $P\omega$, this set will be embedded in $P\omega^2$ in two different ways.

2.1 Definition

(i) Let $A \in P\omega$. Then

$$A' =: < A, \phi > \text{ and } A^* = < A, \overline{A} > .$$

(ii) $(P\omega,')$ is the space $P\omega$ with the Scott topology (see e.q. [1], p. 10). $(P\omega,^*)$ is the space $P\omega$ with the Cantor topology (see e.q. [6], p. 270).

Δ and $\bullet$ will range over the set $\{', *\}$. $P\omega^\Delta$ is the subspace of $P\omega^2$ (with the Scott topology) consisting of the image of $P\omega$ under the map Δ. Note that $\Delta:(P\omega, \Delta) \rightarrow P\omega^\Delta$ is a homeomorphism. A partial map $\Phi: X \rightarrow Y$ on topological spaces X, Y is called continuous if $\Phi|\ Dom(\Phi)$ is continuous on the subspace $Dom(\Phi)$.

2.2 Definition Let $f: P\omega^2 \rightarrow P\omega^2$ be given. The *partial* $\Delta\,\bullet$*-operator induced by* f (notation $\Phi_f^{\Delta\bullet}$) is defined as follows.

$$\Phi_f^{\Delta\bullet}(A)\downarrow \; \Leftrightarrow \; f(A^\Delta) \in P\omega^\bullet;$$

$$(\Phi_f^{\Delta\bullet}(A))^\bullet = f(A^\Delta).$$

That is $\Phi_f^{\Delta\bullet} = \bullet^{-\prime} \circ f \circ \Delta$:

$$\begin{array}{ccc} & \Phi_f^{\Delta\bullet} & \\ P\omega & \rightarrow & P\omega \\ \downarrow\Delta & & \downarrow\bullet \\ P\omega^2 & \rightarrow & P\omega^2 \\ & f & \end{array}$$

If $c \in P\omega^2$, write $\Phi_c^{\Delta\bullet} = \Phi_f^{\Delta\bullet}$ with $f(a) = c \cdot a$ for $a \in P\omega^2$.

2.3 Lemma A partial map $\Phi:(P\omega, \Delta) \rightarrow (P\omega, \bullet)$ is continuous iff Φ is an induced $\Delta, \bullet$ operator by some continuous $f: P\omega \rightarrow P\omega$.

Proof ($\Leftarrow$) $\Phi = \Phi_f^{\Delta\bullet} = \bullet^{-1} \circ f \circ \Delta$ and we are done.

($\Rightarrow$) Define $f_0 = \bullet \circ \Phi \circ \Delta^{-1}: P\omega^2 \rightarrow P\omega^2$. Then f_0 is a partial continuous map. Since $P\omega^2$ is an injective topological space (it is an algebraic, hence continuous lattice, see [7]), f_0 can be extended to a total continuous f. Then $\Phi = \Phi_f^{\Delta\bullet}$. •

Write $\mathfrak{C}_2 = \{f: P\omega^2 \rightarrow P\omega^2 \mid f \text{ computable}\}$.

2.4 Definition Let $\Phi: P\omega \to P\omega$.

(i) Φ is a *partial strong operator* ($\Phi \in \mathcal{C}_s^P$) if $\exists f \in \mathcal{C}_2 \; \Phi = \Phi_f^{\prime *}$

(ii) Φ is a *partial Turing operator* ($\Phi \in \mathcal{C}_T^P$) if $\exists f \in \mathcal{C}_2 \; \Phi = \Phi_f^{**}$

(iii) Φ is a *partial enumeration operator* ($\Phi \in \mathcal{C}_e^P$) if $\exists f \in \mathcal{C}_2 \; \Phi = \Phi_f^{\prime\prime}$

(iv) Φ is a *partial weak operator* ($\Phi \in \mathcal{C}_w^P$) if $\exists f \in \mathcal{C}_2 \; \Phi = \Phi_f^{*\prime}$.

$$\text{Write } \mathcal{C}_x = \{f \in \mathcal{C}_x^P \mid f \text{ is total}\} \text{ for } x \in \{s, T, e, w\}.$$

Example The jump operator $\Phi(A) = A^j = \{x \mid \varphi_x^A(x)\downarrow\}$ is a partial weak operator. Namely define

$$c_- = \{((n,m),p) \mid \exists q \, (p,q,n,m) \in W_{p(p)}\}.$$
$$c_+ = \phi.$$

then $\Phi = \Phi_c^{*\prime}$; see [6] p. 132 for the definition of $W_{p(p)}$.

2.5 Definition

(i) Let D be some class of partial operators and $A, B \in P\omega$. A is *D-reducible* to B (notation $A \leq_D B$) if $\exists \Phi \in D \;\; \Phi(B) = A$.

(ii) A is *strongly* reducible to B (notation $A \leq_s B$) if $A \leq_{\mathcal{C}_s^P} B$;
A is *Turing* reducible to B (notation $A \leq_T B$) if $A \leq_{\mathcal{C}_T^P} B$;
A is *enumeration* reducible to B (notation $A \leq_e B$) if $A \leq_{\mathcal{C}_e^P} B$
A is *weakly* reducible to B (notation $A \leq_w B$) if $A \leq_{\mathcal{C}_w^P} B$.

For $a, b, \in P\omega^2$ write $a \leq b$ if $\exists c \in P\omega_c^2 \; a = c\, b$. then one has

$$A \leq_s B \iff A^* \leq B'$$

$$A \leq_T B \iff A^* \leq B^*$$

$$A \leq_e B \iff A' \leq B'$$

$$A \leq_w B \iff A' \leq B^*$$

2.6 Proposition

(i) Any partial strong operator can be extended to a total enumeration operator. (notation: $\mathfrak{C}_s^P \rightsquigarrow \mathfrak{C}_e$).

(ii) $\mathfrak{C}_T^P \rightsquigarrow \mathfrak{C}_w$

(iii) $\mathfrak{C}_e^P \rightsquigarrow \mathfrak{C}_e$

(iv) $\mathfrak{C}_\omega^P \rightsquigarrow \mathfrak{C}_w$

(v) $\mathfrak{C}_s^P \subseteq \mathfrak{C}_T^P$

(vi) $\mathfrak{C}_e^P \subseteq \mathfrak{C}_w^P$

Proof Define $i{:}P\omega^2 \to P\omega^2$ by $i(<A,B>) = <A,\phi>$. Clearly i is definable.

(i) Note that $\Phi'^{*}_f \subseteq \Phi''_{i\circ f}$, since $i\circ{*} = {'}$ and this last operator is total ($i(P\omega^2) = P\omega'$):

$$\begin{tikzcd}
P\omega \arrow[r, "\Phi'^{*}_f"] \arrow[d, "'"] & P\omega \arrow[dr, "'"] & \\
P\omega^2 \arrow[r, "f"'] & P\omega^2 \arrow[r, "i"'] \arrow[u, phantom, "*"] & P\omega^2
\end{tikzcd}$$

(ii) Similarly $\Phi^{**}_f \subseteq \Phi^{*\prime}_{i\circ f}$.

(iii) Now $\Phi''_f \subseteq \Phi''_{i\circ f}$, since $i\circ{'} = {'}$

(iv) Similarly $\Phi^{*\prime}_f \subseteq \Phi^{*\prime}_{i\circ f}$.

(v) Now $\Phi'^{*}_f = \Phi^{**}_{f\circ i}$, since $i\circ{*} = {'}$

(vi) Similarly $\Phi''_f = \Phi^{*\prime}_{f\circ i}$. •

2.7 Corollary

$$\begin{array}{ccc}
A \leq_s B & \Rightarrow & A \leq_T B \\
\Downarrow & & \Downarrow \\
A \leq_e B & \Rightarrow & A \leq_w B.\ \bullet
\end{array}$$

It is not true that $\mathfrak{C}_T^P \rightsquigarrow \mathfrak{C}_T$ or $\mathfrak{C}_s^P \rightsquigarrow \mathfrak{C}_s$, see 2.14 and 2.16 below.

The classes $\mathfrak{C}_e$, $\mathfrak{C}_w$ and $\mathfrak{C}_T^P$ turn out to consist of known recursion theoretic operators.

2.8 Theorem $\Phi \in \mathfrak{C}_e$ iff Φ is an enumeration operator as defined in [6], p. 147.

Proof ($\Leftarrow$) By definition $\Phi(B) = F{\cdot}B$ for some $F \in P\omega_c = \mathfrak{R}E$. Define $b = <\{((n,o),m) \mid (n,m) \in F\}, \phi>$. Then $b \in P\omega^2_{\mathfrak{C}}$ and $\Phi = \Phi''_b$.

($\Rightarrow$) Let $\Phi = \Phi''_b$ be total and $b \in P\omega^2_c$. Define $F = \{(n,m) \mid ((n,o),m) \in b\_\} \in P\omega_c$. Then $\Phi(B) = F \cdot B$ for all $B \in P\omega$. •

In order to describe weak and partial Turing operators, two Lemmas are needed.

2.9 Lemma

(i) There is a recursive function g such that for all $i \in \mathbf{N}$ and $A, B \in P\omega$

$$\Phi^{**}_{\omega_i}(B) = A \quad \Leftrightarrow \quad c_A = \varphi^{B}_{g(i)}$$

(ii) There is a recursive function h such that for $i \in \mathbf{N}$ with $\Phi^{*\prime}_{\omega_i}$ total and all $A, B \in P\omega$

$$\Phi^{*\prime}_{\omega_i}(B) = A \quad \Leftrightarrow \quad A = W^{B}_{h(i)}$$

Proof

(i) Define

$$\psi^{B}(i,m) = \begin{cases} 1 & \text{if } \exists e_n \sqsubseteq B^* \ (n,m) \in \omega_{i-}; \\ 0 & \text{if } \exists e_n \sqsubseteq B^* \ (n,m) \in \omega_{i+}; \\ \uparrow & \text{else.} \end{cases}$$

By the relativised $s-m-n$ theorem $\psi^{b}(i,m) = \varphi^{B}_{g(i)}(m)$ for some recursive g. This g works. (Note that if $\omega_i\, B^* \in P\omega^*$, then $\neg\exists m\ \exists e_n \sqsubseteq B^*\ (n,m) \in \omega_{i-} \cap \omega_{i+}$).

(ii) Similarly let h be a recursive function such that

$$\varphi^{B}_{h(i)}(m) = \chi^{B}(i,m) = \begin{cases} 1 & \text{if } \exists\, e_n \sqsubseteq B^* \ (n,m) \in \omega_{i-}; \\ \uparrow & \text{else.} \end{cases}$$

Then h works. •

2.10 Lemma

(i) There is a recursive function g such that for all $i \in \mathbf{N}$ and all $A, B \in P\omega$

$$c_A = \varphi^{B}_{i} \quad \Leftrightarrow \quad \Phi^{**}_{g(i)}(B) = A$$

(ii) There is a recursive function h such that for all $i \in \mathbf{N}$ and all $A, B \in P\omega$

$$A = W^{B}_{i} \quad \Leftrightarrow \quad \Phi^{*\prime}_{h(i)}(B) = A$$

Proof

(i) Given any regular r.e. set $W_{\rho(i)}$ cf. [6], p. 132, define

$$a = < \{((p,q),m) \mid (m,o,p,q) \in W_{\rho(i)}\}, \{((p,q),m) \mid (m,1,p,q) \in W_{\rho(i)}\} > .$$

Clearly $a \in P\omega_c^2$ and an index for a is uniformly effective in i. Moreover $c_A = \varphi_i^B$ iff $A^* = a\,B^*$ for all A, B

(ii) Similarly with

$$a = < \{((p,q),m) \mid \exists n(m,n,p,q) \in W_{\rho(i)}\}, \phi > . \bullet$$

From 2.9 and 2.10 one obtains the following.

2.11 Theorem

(i) $\mathfrak{C}_T^P = \{\Psi_0, \Psi_1, ...\}$, where

$$\Psi_i(A) = \begin{cases} B & \text{if } c_B = \varphi_i^A; \\ \uparrow & \text{else.} \end{cases}$$

(ii) $\mathfrak{C}_w = \{\Gamma_0, \Gamma_1, ...\}$, where $\Gamma_i(A) = W_i^A$. $\bullet$

Now the reducibility notions can be characterized.

2.12 Theorem Let $A, B \in P\omega$. Then

(i) $A \leq_e B \Leftrightarrow A$ is enumeration reducible to B, cf. [6] p. 146;
(ii) $A \leq_s B \Leftrightarrow A \leq_e B$ and $\overline{A} \leq_e B$;
(iii) $A \leq_T B \Leftrightarrow A$ is recursive in B;
(iv) $A \leq_w B \Leftrightarrow A$ is r.e. in B, cf. [6] p. 133.

Proof

(i) By 2.8.
(ii) ($\Leftarrow$) Let $F, G \in \mathfrak{R}E$ be such that $A = FB$ and $\overline{A} = GB$. Define

$$a = < \{((n,o),m) \mid (n,m) \in F\}, \{((n,o),m) \mid (n,m) \in G\} >$$

Then $a \in P\omega_c^2$ and $\Phi_a'^*(B) = A$.

($\Rightarrow$) Let $\Phi_a'^*(B) = A$. Define $F = \{(n,m) \mid ((n,o),m) \in a_-\}$ and $G = \{(n,m) \mid ((n,o),m) \in a_+\}$. Then $A = FB, \overline{A} = GB$.

(iii) By 2.11(i).
(iv) By 2.11(ii). $\bullet$

Now it is shown why partial Turing and strong operators cannot always be made total.

2.13 Lemma Let $\Phi \in \mathfrak{C}_s^P$ and $\phi \in Dom\Phi$. Then for all $B \in Dom\Phi$ one has $\Phi(B) = \Phi(\phi)$. Moreover $\Phi(\phi)$ is recursive.

Proof First note that $A^* \sqsubseteq B^* \Rightarrow A = B$. Let $\Phi = \Phi_f'^*$, i.e. $\Phi(A)^* = f(A')$ for $A \in dom\Phi$. Then by monotonicity

$$\Phi(\phi)^* = f(<\phi, \phi>) \sqsubseteq f(<B, \phi>) = \Phi(B)^*$$

for $B \in Dom\Phi$. *Hence* $\Phi(B) = \Phi(\phi)$ on $Dom\Phi$. Moreover $\Phi(\phi)^* =< \Phi(\phi), \overline{\Phi(\phi)} > \in P\omega_c^2 = \Re E^2$, since f if computable. Hence $\Phi(\phi)$ is recursive. •

2.14 Corollary $\mathfrak{C}_S^P \not\rightsquigarrow \mathfrak{C}_S$.

Proof Let K be a non recursive r.e. set. Note that $K \leq_e \overline{K}$ and $\overline{K} \leq_e K$. Hence by 2.12(ii) one has $K \leq_s \overline{K}$, i.e. $\Phi(\overline{K}) = K$ with $\Phi \in \mathfrak{C}_S^P$. By 2.13 Φ cannot be made total. •

2.15 Theorem (Nerode). Let $\leq_{tt}$ denote truth table reducibility, cf. [6], p. 110. Then for all $A, B \in P\omega$

$$A \leq_{tt} B \iff \exists \Phi \in \mathfrak{C}_T \ \Phi(B) = A$$

For a proof, see [6], th.9 XIX. The idea is that $(P\omega, *)$ is a compact metric space, hence a continuous Φ on it is uniformly continuous. This provides the required (effectively uniformly bounded) truth table conditions.

While $(P\omega, *)(\simeq 2^{\mathbb{N}})$ is an injective space, see [7]; therefore all partial functions on it can be extended to total ones. However, the extension may fail to be computable.

2.16 Corollary $\mathfrak{C}_T^P \not\rightsquigarrow \mathfrak{C}_T$.

Proof By 2.15, 2.12(iii) and the fact that $\leq_T \not\Rightarrow \leq_{tt}$, cf [6], cor.9 XVIII. •

A concrete example of a partial Turing operator that cannot be made total is the following. Define

$$\Phi(A) = \{q - p\} \text{ if } p, q \text{ are the first two elements of } A$$
$$\uparrow \text{ if } A \text{ has at most one element.}$$

By Church thesis and 2.11 Φ is a partial Turing operator. Φ cannot be extended to a total Turing operator $\Phi^\sim$ because, by the compactness of $(P\omega, *)$, $\Phi^\sim$ has to be uniformly continuous, which is impossible.

§3. The Turing-Rogers Operators

In [6] another class $\mathfrak{C}_{TR}^P$ of partial operators is suggested. It will be shown that $\mathfrak{C}_{TR}^P = \mathfrak{C}_T^P$.

3.1 Definition Let X, Y be sets and let $i{:}X \to Y$ be an injective map. Let $g{:}Y \to Y$. Then $f{:}X \to X$ is **defined by** g **via** i if $f = i^{-\prime} \circ g \circ i$ with $Dom(f) = \{x \mid g(i(x)) \in i(X)\}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "i"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

3.2 Notation

(i) $\mathcal{P} = \mathbf{N} \to \mathbf{N}$; $\mathcal{P}_{01} = \mathbf{N} \to \{0,1\}$; $\mathcal{PR} = \{\varphi \in \mathcal{P} \mid \varphi$ is partial recursive$\}$.

(ii) $\tau{:}\mathcal{P} \to P\omega$ is defined by

$$\tau(\varphi) = \{(n, m) \mid \varphi(n) = m\}.$$

(iii) $c{:}P\omega \to \mathcal{P}_{01}$ is defined by

$$c_A = c(A) = \text{ characteristic function of } A (\text{equals } 0 \text{ if argument in } A).$$

3.3 Definition

(i) $\Phi{:}\mathcal{P} \to \mathcal{P}$ is a **partial recursive operator**, notation $\Phi \in \mathcal{C}_r^P$ if Φ is defined by some total $\Psi \in \mathcal{C}_e$ via $\tau{:}\mathcal{P} \to P\omega$.

(ii) $\Phi{:}P\omega \to P\omega$ is a **partial Turing-Rogers operator**, notation $\Phi \in \mathcal{C}_{TR}^P$, if Φ is defined by some total $\Psi \in \mathcal{C}_r$ via $c{:}P\omega \to \mathcal{P}$.

3.4 Lemma Let $g{:}P\omega^2 \to P\omega^2$ be computable such that $g(T^\omega) \subseteq T^\omega$. Then $g \mid T^\omega$ is computable in T^ω.

Proof Let $f = g \mid T^\omega$. f is continuous since T^ω is a subspace of $P\omega^2$. An easy computation shows that if $a = graph(f)$ as defined for T^ω, then $a \in T_c^\omega$. •

Now we need yet another characterization of $\mathcal{C}_T^P$.

3.5 Proposition $\Phi \in \mathcal{C}_T^P$ iff Φ is defined by some computable $f{:}T^\omega \to T^\omega$ via $*{:}P\omega \to T^\omega$.

Proof ($\Rightarrow$) By 2.9(i) there is an index i such that for all $A \in P\omega$

$$c(\Phi(A)) = \varphi_i^A.$$

Define $d = \langle d_-, d_+ \rangle$ with

$$d_- = \{((p,q), m) \mid (m, o, p, q) \in W_{\rho(i)}\},$$
$$d_+ = \{((p,q), m) \mid (m, l, p, q) \in W_{\rho(i)}\}.$$

where $W_{p(i)}$ is the "regularization" of W_i as defined in [6], p. 132. Define $g(a) = da$ in $P\omega^2$. Clearly g is computable and Φ is defined by g via $* : P\omega \to P\omega^2$. By the regularity of $W_{p(i)}$ it follows that

$$\forall a \in T^\omega \quad g(a) \in T^\omega.$$

By 3.4 $f = g \mid T^\omega$ is computable. Moreover Φ is defined by f via $*$:$P\omega \to T^\omega$.

($\Leftarrow$) Let f:$T^\omega \to T^\omega$ be computable. By 1.9 f can be extended to a computable $f^\sim$:$P\omega^2 \to P\omega^2$. Then Φ defined by f via $*$ is also defined by $f^\sim$ via $*$, i.e. $\Phi \in \mathfrak{C}^P_T$. •

Remark Similar results hold for the classes $\mathfrak{C}^P_c$ and $\mathfrak{C}^P_w$. However not for the strong operators: the only partial strong operators defined via T^ω are the constant ones.

3.6 Lemma

(i) Define $SG : \mathcal{P} \to \mathcal{P}$ by $SG(\psi) = sg \cdot \psi$. Then $SG \in \mathfrak{C}_r$, $SG(\mathcal{P}) \subseteq \mathcal{P}_{01}$ and $\forall \psi \in \mathcal{P}_{01}$ $SG(\psi) = \psi$.

(ii) If $\Phi \in \mathfrak{C}^P_{TR}$, then it may be assumed that Φ is defined by a $\Psi \in \mathfrak{C}^P_r$ with $\Psi(\mathcal{P}) \subseteq \mathcal{P}_{01}$.

Proof

(i) Let $A = \{(n, (p, sg(q))) \mid E_n = \{(p, q)\}\}$ and $\Phi_e(B) = A \cdot B$ defined in $P\omega$. Then $\Phi_e \in \mathfrak{C}_c$ and SG is defined by Φ_e via τ, i.e. $\Phi \in \mathfrak{C}_r$. The rest is clear.

(ii) By (i). •

Let $\sigma : T^\omega \to \mathcal{P}$ be defined by

$$\sigma(< A, B >)(n) = \begin{cases} 0 \text{ if } n \in A; \\ 1 \text{ if } n \in B; \\ \uparrow \text{ else.} \end{cases}$$

That is $\sigma(a)$ is the partial characteristic map of a.

3.7 Lemma Let f: $T^\omega \to T^\omega$. Then f is computable iff f is defined via σ by a total $\Phi \in \mathfrak{C}_r$ with $\Phi(\mathcal{P}) \subseteq \mathcal{P}_{01}$.

Proof ($\Rightarrow$) Take $\chi = \tau \circ \sigma$ and let h, ℓ be recursive functions such that $e_{h(n)} = b_n$ and $E_{\ell(h)} = \chi(b_n)$. Define

$$D = \{(\ell(n), (m, i)) \mid ((-h(n); m) \in \lambda x \cdot f(x)\_\wedge i = 0) \vee ((+h(n); m) \in \lambda x \cdot f(x)\_\wedge i = 1)\}.$$

Then $D \in P\omega_c$, hence $\Psi = \lambda A \cdot DA \in \mathfrak{C}_c$. An easy computation shows that f is defined by Ψ via χ (use $e_{h(n)} \sqsubseteq a$ iff $E_{\ell(n)} \subseteq \chi(a)$).

$$\begin{tikzcd}
& T^\omega \arrow[rr, "f"] \arrow[dl, "\sigma"'] \arrow[dd, "\chi"] & & T^\omega \arrow[dl, "\sigma"'] \arrow[dd, "\chi"] \\
\mathcal{P} \arrow[rr, dashed, "\Phi"] \arrow[dr, "\tau"'] & & \mathcal{P} \arrow[dr, "\tau"'] & \\
& P\omega \arrow[rr, dashed, "\Psi"'] & & P\omega
\end{tikzcd}$$

Let $\Phi \in \mathfrak{C}^P_r$ be defined by Ψ via τ. Since by definition $\Psi(\tau(\mathcal{P})) \subseteq \tau(\mathfrak{R}_{01})$, it follows that Φ is total and $\Phi(\mathcal{P}) \subseteq \mathfrak{R}_{01}$.

($\Leftarrow$) Let $f = \sigma^{-1} \circ \Phi \circ \sigma = \sigma^{-1} \circ SG \circ \Phi \circ \sigma$. By it suffices to show that $f_k = f \mid T^\omega_c$ is computable. But f_k is the composition of the morphisms $\sigma \mid T^\omega_c, \Phi \mid \mathcal{P}\mathfrak{R}$ and $\sigma^{-1} \circ SG \mid \mathcal{P}R$, hence itself a morphism. Therefore we are done by the generalized Myhill-Shepherdson theorem 1.8. •

3.8 Theorem $\mathfrak{C}^P_{TR} = \mathfrak{C}^P_T$.

Proof ($\subseteq$) Let Φ be defined by $\Psi \in \mathfrak{C}_r$ via c.

$$\begin{tikzcd}
& P\omega \arrow[rr, "\Phi"] \arrow[dl, "*"'] \arrow[dd, "c"] & & P\omega \arrow[dl, "*"'] \arrow[dd, "c"] \\
T\omega \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & T\omega \arrow[dr, "\sigma"'] & \\
& \mathcal{P} \arrow[rr, "\Psi"'] & & \mathcal{P}
\end{tikzcd}$$

By 3.6(ii) it may be assumed that $\Phi(\mathcal{P}) \subseteq \mathfrak{R}_{01}$. Define f: $T^\omega \to T^\omega$ by Φ via σ. Then f is computable by 3.7. By a diagram chase, one sees that Φ is defined by f via $*$.

($\supseteq$) By an even simpler diagram chase, using also 3.5. •

Question Can the Kreisel-Lacombe-Shoenfield theorem, cf. [6] p. 362, be proved by the methods of this paper?

§4. Interpolation

Given finitely many distinct elements $B_0, \ldots, B_p \in P\omega$, then for each $A_0, \ldots, A_p \in P\omega$ there is a total Turing operator Φ such that $\Phi(B_i) = A_i$, $o \leq i \leq p$, provided that each B_i can be mapped onto A_i at all (i.e. $A_i \leq_u B_i$ for $o \leq i \leq p$).

4.1 Interpolation Theorem Let $B_o, \ldots, B_p$ be a collection of pairwise different sets. Assume

$$A_i \leq_{tt} B_i \text{ via } f_i, \quad \text{for } o \leq i \leq p.$$

Then $\exists \Phi \in \mathcal{C}_T \ \ \forall i \leq p \ \ \Phi(B_i) = A_i$.

[In classical notation, for distinct B_i's, $i = 1, \ldots, p$:

$\forall i \leq p \ \exists z (C_{A_i} = \varphi_z^{B_i} \wedge \forall B \ \varphi_z^B$ is a characteristic function)

implies

$\exists z \ \forall i \leq p(\ldots idem \ldots)]$.

Proof Since $(P\omega, *)$ is an Hausdorf space there are disjoint clopen neighborhoods $\mathcal{A}_{n_i} = \{a \in P\omega^2 \mid e_{n_i} \sqsubseteq a\}$ such that $B_i \in \mathcal{A}_{n_i}$ for $o \leq i \leq p$.

$$\text{Let } \mathcal{A} = \bigcup_{i \in p} \mathcal{A}_{n_i}.$$

Note that $\overline{\mathcal{A}}$ is also open and $\overline{\mathcal{A}} = \{A \mid \forall i \leq p (A \cap (e_{n_i})_+ \neq \phi) \vee (\overline{A} \cap (e_{n_i})_- \neq \phi)\}$.

Let $f_i(q)$ be (the index of) the tt-condition $<< m_1, \ldots, m_{k_i} >, a_i^q >$. Let $\vec{j}$ range over $\{0,1\}^{k_i}$. Define

$$e^{i,q}(\vec{j}) = e_{n_i} \cup < \{m_h \mid h \leq k_i \ \wedge \ j_h = 1\}, \{m_h \mid h \leq k_i \wedge j_h = 0\} > .$$

Note that

$$e^{h,q}(\vec{j}) \sqsubseteq B_i^* \Rightarrow h = i. \tag{1}$$

Finally define

$$c_- = \{(m,q) \mid \exists i \leq p \ \exists \vec{j} \in \{0,1\}^{k_i} \ e_m = e^{i,q}(\vec{j}) \wedge a_i^q(\vec{j}) = 1\}$$

$$c_+ = \{(m,q) \mid \exists i \leq p \ \exists \vec{j} \in \{0,1\}^{k_i} \ e_m = e^{i,q}(\vec{j}) \wedge a_i^q(\vec{j}) = 0\} \ \cup \ D$$

where $D = \{(m,q) \mid \forall i \leq p(((e_m)_- \cap (e_{n_i})_+ \neq \phi) \vee ((e_m)_+ \cap (e_{n_i})_- \neq \phi)) \wedge q \in \omega\}$.

Claim 1 $A_i^* = c\,B_i^*$ for $o \leq i \leq p$. Indeed

$$q \in (c\,B_i^*)_- \Leftrightarrow \exists e_m \sqsubseteq B_i^*\ (m, q) \in c_-$$

$$\Leftrightarrow \exists e_m \sqsubseteq B_i^*\ \exists h \leq p\ \exists \vec{j}\ e_m = e^{h,q}(\vec{j}) \wedge \alpha_h^q(\vec{j}) = 1$$

$$\Leftrightarrow \exists\, \vec{j}\ e^{i,q}(\vec{j}) \sqsubseteq B_i^* \wedge \alpha_i^q(\vec{j}) = 1 \qquad (by\ (1))$$

$$\Leftrightarrow B_i \text{ satisfies the } tt\text{-condition } f_i(q)$$

$$\Leftrightarrow q \in A_i$$

Similarly $(c\,B_i^*)_+ = \overline{A}_i$, since for no (m, q) one has $e_m \sqsubseteq B_i^* \wedge (m, q) \in D$ (because $e_{n_i} \sqsubseteq B_i^*$).

Claim 2 $\forall B \in P\omega \quad c\,B^* \in P\omega^*$

Case 1 $B \in \mathcal{A}$. Then $e_{n_i} \sqsubseteq B^*$ for some $i \leq p$, hence

$$\forall q\ \exists! \vec{j} \in \{0, 1\}^{k_i}\ \ e^{i,q}(\vec{j}) \sqsubseteq B^*.$$

Now if $\alpha_i^q(\vec{j}) = 1$ then $q \in (c\,B^*)_-$ else $q \in (c\,B^*)_+$. So $(c\,B^*)_- \cup (c\,B^*)_+ = \omega$.
If $q \in (c\,B^*)_- \cap (c\,B^*)_+$ then $\alpha_i^q(\vec{j}) = 1 \wedge \alpha_i^q(\vec{j}) = 0$, a contradiction. Thus $c\,B^* \in P\omega^*$.

Case 2 $B \in \overline{\mathcal{A}}$. Then by the definition of D it easily follows that $c\,B^* = <\phi,\ \omega> \in P\omega^*$. •

4.2 Remarks

(i) By an even simpler technique one can also show that if $\{B_i\}_{i \in \mathrm{N}}$ is a set of isolated elements in $(P\omega, *)$ and for some recursive f

$$\forall i\ \exists k\ \ A_i^* = \omega_{f(k)}\, B_i^*,$$

then for some

$$\Phi \in \mathbb{C}_T^P$$

$$\forall i\ \ A_i = \Phi(B_i).$$

Moreover one may assume that $dom(\phi)$ is not meager. [By assumption $\exists h' \forall i\ B_i \notin \mathcal{A}_{h'}$; let $B_i \in \mathcal{A}_{n_i} \subseteq \overline{\mathcal{A}}_{h'}$ for all i - this is possible since $\mathcal{A}_{h'}$ is also closed. Then the following $a \in P\omega_c^2$ will do the job:

$$a_- = \{(m,p) \mid \exists i\ \exists q((q,p) \in (\omega_{f(i)})_- \wedge e_m = e_q \cup e_{n_i}) \vee (m = h' \wedge p \in \mathbf{N})\}$$

$$a_+ = \{(m,p) \mid \exists i\ \exists q((q,p) \in (\omega_{f(i)})_+ \wedge e_m = e_q \cup e_{n_i}\}.$$

The last clause in the definition of a_- gives the non meagerness of $dom(\Phi)$, making Φ defined (equal ω) on $\mathcal{A}_{h'}$.]

(ii) In the same way as in (i), under similar assumptions, one can find an interpolating $\Phi \in \mathcal{C}_w^P$. By 2.6(iv), Φ may actually be taken in $\mathcal{C}_w$.

(iii) It is not difficult to see that 4.1 cannot be extended to a result as in (i). [Take the B_i a converging sequence and the $A_i(\leq_{tt} B_i)$ not converging.] Also (i) cannot be strengthened by dropping the isolatedness or the uniformity.

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