A NOTE ON EQUIVALENCES
AMONG LOGICS OF PROGRAMS

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Abstract. Several different first order formal logics of programs--Algorithmic Logic, Dynamic Logic, and Logic of Effective Definitions--are compared and shown to be equivalent to a fragment of constructive $L_{\omega_1\omega}$. When programs are modelled as effective flowcharts, the logics of deterministic and nondeterministic programs are equivalent.

1 Introduction

A number of systems of formal logics which extend predicate calculus have been proposed for reasoning about sequential and nondeterministic programs. These include in rough chronological order

1. The infinitary logic $L_{\omega_1\omega}$ -- suggested by ENGELER 67 as a logic for programming,

2. Algorithmic Logic (AL) -- defined and developed by SALWICKI, et.al. 70,

3. $\mu$-calculus -- defined by HITCHCOCK and PARK 73; extended by DE BAKKER 80,

4. Dynamic Logic (DL) -- PRATT 76,

5. Programming Logic (PL) -- CONSTABLE and O'DONNELL 78,


Each of these logical systems actually represents a family of formal logics, instances of the family being determined by the choice of a few parameters. The principal parameter is the class of programs allowed in formulas. For example, in the case of DL some variants which have been considered are

- **regular** DL, in which programs are taken essentially to be finite, possibly nondeterministic, flowchart schemes with atomic formulas as tests and with *simple assignment statements* of the form $x := \tau$ where $\tau$ is a term,

- **regular-array** DL in which *array assignments* of the form $\tau_1 = \tau_2$ may also occur (cf. MEYER and WINKLMANN 80),
- regular $DL^+$ in which for every finite flowchart $\alpha$, the predicate $LOOPS_\alpha$, which asserts that $\alpha$ has an infinite computation, is included as an extra atomic formula (cf. MEYER and WINKLMANN 80).\textsuperscript{2}

- recursive-call $DL$ in which programs are taken to be flowchart schemes containing recursive calls with arguments (cf. GREIBACH 75, DE BAKKER 80).

In general, such different choices of the parameters lead to logics which differ in expressive power. For example, TIURYN 81 has recently shown that there is a formula of recursive-call $DL$, as well as one of regular-array $DL$, which is not equivalent to any formula of regular $DL$. On the other hand, MEYER and WINKLMANN 80 have shown that regular $DL$ and regular $DL^+$ are equivalent in expressive power. MEYER and PARIKH 81 have also demonstrated distinctions among the expressive powers of several other versions of $DL$ and $L^{\omega_1 \omega}$.

Thus there are genuine distinctions in the expressive, and also model theoretic and undecidability properties among the various instances of $DL$. These distinctions complicate the problem of comparing the six systems of programming logics listed above. For example, the bulk of the literature on AL defined that system in the particular version where programs are deterministic while schemes.\textsuperscript{3} Since the original $DL$ allowed nondeterministic schemes, it appeared that $DL$ and AL represented genuinely distinct conceptions of programming logic.

Nevertheless, we claim that with appropriately matched parameters, $DL$, AL, and LED, are actually equivalent systems. We believe that PL can be incorporated into this common framework as well, although its numerous "practical" features make it harder to grasp theoretically.

\textsuperscript{2}However, $LOOPS_\alpha$ may not occur as a test in a program.

\textsuperscript{3}Only recently has an AL with nondeterministic schemes been considered by MIRKOWSKA 80.
These systems can be described in more classical terminology as fragments of the constructive portion of $L_{\omega_1^{\omega}}$, with the different instances of the systems characterized by various simple syntactic conditions on infinitary formulas. Thus, we argue that there is a common intuition which leads to the DL-AL-LED-PL framework for programming logic. In what follows we focus on this framework.\(^4\)

In order to compare the DL-AL-LED-PL frameworks, we restrict ourselves to instances of these systems using what we regard as the mathematically most natural and robust notion of *computability* over arbitrary structures, namely computability by *effective flowcharts*. Effective flowcharts may be described informally as generally infinite, nondeterministic, uninterpreted flowchart schemes whose basic instructions are assignment statements and whose basic tests consist of atomic formulas (including equations). Moreover, given a box of the flowchart, one can effectively find the instruction in that box, the number of edges leaving the box, and the endpoints of those edges. For technical convenience we require that the *signature* (i.e., set of symbols occurring, including variables) of any flowchart is *finite*.

A *state* provides an interpretation for all function, predicate, and variable symbols. Given a state, a nondeterministic flowchart defines a set of executable instruction sequences. The set of states in which execution of these instruction sequences can finally terminate is the set of *output states* for the given input state. Thus, any flowchart $\alpha$ defines a binary *input-output* relation $R_\alpha$ on states where

$$R_\alpha = \{(s,t) | \text{starting in state } s, \text{ there is an executable sequence of instructions in } \alpha \text{ which finishes in output state } t\}.$$ 

\(^4\)Technical results of PARK 76 for $\mu$-calculus, and MEYER and PARikh 81 for the constructive fragment of $L_{\omega_1^{\omega}}$ show that these latter logics are incomparable in expressive power, and both are strictly greater in expressive power than logics in the DL-AL-LED-PL framework unless the notion of program scheme is stretched unreasonably.
If there is an infinite executable sequence starting in state \( s \), then \( \alpha \) is said to loop from state \( s \). Formal definitions are available in MEYER and WINKLMANN 80, MEYER and HALPERN 80, MEYER and PARIKH 80, TIURYN 80.

Friedman, cf. SHEPHERDSON 73, proposed a notion of effective definitional scheme as the most general model of effective computability in arbitrary structures. These may be described as the special case of effective flowcharts which are of the form

\[
\text{if } P_1 \text{ then } \text{ASSIGN}_1 \text{ else if } P_2 \text{ then } \text{ASSIGN}_2 \text{ else if } P_3 \ldots
\]

where \( P_i \) is a finite conjunction of atomic formulas or their negations, and \( \text{ASSIGN}_i \) is a sequence of assignment statements of the form \( x := \tau \) with distinct variables \( x \) on the lefthand side of each statement in the sequence.

We can generalize effective definitional schemes to be nondeterministic. These nondeterministic effective definitional schemes can be informally described as the infinite parallel OR of statements of the form

\[
\text{if } P_i \text{ then } \text{ASSIGN}_i \text{ else ABORT fi,}
\]

where ABORT is a program with empty input-output relation, e.g., while true do anything od. Equivalent notions of universal classes of effective procedures on arbitrary structures have been proposed by many other researchers. In particular, it is easy to show

**Lemma 1:** The following classes of program schemes define the same class of input-output relations:

1. (Non)Deterministic effective flowcharts without array assignments (i.e., simple assignments only),

2. (Nondeterministic) Effective definitional schemes,

3. (Non)Deterministic finite flowcharts without array assignments but with stacks.
Similar definitions and lemma can be given for the case that array assignments are allowed. These results indicate the invariance of the class of computable input-output relations between states defined by effective flowcharts.

Our main observation is that when effective flowcharts are taken as the notion of program in the programming logics listed above, then all can be reduced to a simple fragment of constructive $L_{\omega_1\omega}$ which we define next.

**Definition 2:** Let $L_{\omega_1\omega}$ be the class of infinitary first order formulas defined inductively as follows:

(a) if $P_1, P_2, \ldots$ is a recursively enumerable sequence of quantifier-free formulas of predicate calculus among which there are only finitely many free variables, then $\forall\{P_i|i\leq 1\}$ is a basic formula of $L_{\omega_1\omega}$.

(b) if $p, q$ are formulas of $L_{\omega_1\omega}$ then so are $\neg p, p \land q, p \lor q, \exists x[p], \forall x[p]$.

**Theorem 3:** There is an effective procedure to translate a formula of any one of the following formal logics into an equivalent formula of any of the others:

1. $L_{\omega_1\omega}$,

2. DL of deterministic effective flowcharts without array assignments (i.e., only simple assignments occur), henceforth called $DDL_{w/o-array}$

3. DL of deterministic effective flowcharts (i.e., array assignments may occur) henceforth called $DDL$,

4. $DL^+$ of nondeterministic effective flowcharts without array assignments, henceforth called $DL^+_{w/o-array}$,

5. LED,

6. Logic of nondeterministic effective definitional schemes (without array assignments),

7. AL of deterministic effective flowcharts without array assignments,
8. AL of nondeterministic effective flowcharts without array assignments and without the iteration quantifier $\cap$.

We would like to emphasize that according to Theorem 3, $\text{DDL-w/o-array}$ and $\text{DL}^+-\text{w/o-array}$ are equivalent, viz., adding nondeterminism to effective flowcharts does not increase the expressive power of the dynamic logic.

Although in many programming situations nondeterminism is a significant addition, we can explain informally why it adds nothing to the logic of deterministic effective schemes: the rich control structure provided by arbitrary effective flowcharts enables a deterministic scheme $\alpha_d$ to "check the results" of any nondeterministic scheme $\alpha$ by carrying out a backtracking search. In particular, suppose $\alpha$ is a nondeterministic effective flowchart without array assignments whose registers, i.e., free variables, are $x = x_0, \ldots, x_{n-1}$. Then there is a deterministic effective flowchart $\alpha_d$ such that $\alpha_d(x, y)$ halts iff $\alpha(x)$ can halt with the final contents of registers $x$ set to $y$. Thus the assertion that after $\alpha(x)$ halts, it is possible that some property $p(x)$ holds, is equivalent to the assertion that there exist $y$ such that $\alpha_d(x, y)$ halts and $p(y)$ holds. In this way, an existentially quantified assertion about a deterministic flowchart has the same expressive power as an assertion about a nondeterministic flowchart.

For more restricted control structures which cannot carry out the backtracking search, nondeterminism indeed makes a difference: P. Berman, J. Halpern, and J. Tiuryn have recently shown that for regular programs, $\text{DDL}$ is strictly less expressive than $\text{DL}$.

In the case that array assignments do occur in nondeterministic programs, our proof of Theorem 3 breaks down. The nondeterministic flowchart $\alpha$ may have registers $x$ and also assignable arrays, i.e., function symbols $f$. Again, there is a deterministic "checking" flowchart $\alpha_d$ such that $\alpha_d(x, f, y, g)$ halts iff $\alpha(x, f)$ can halt with the final values of registers $x$ and arrays $f$ equal to $y, g$. Now, however, in order to reduce an assertion about $\alpha$ to one about $\alpha_d$ as above, it is necessary to existentially bind not only the $y$ variables by also the
function symbols \( g \). This second order quantification exceeds the power of \( DL \). But because the values of the arrays \( g \) differ only finitely from the values of the \( f \), the full power of second order quantification is not necessary. If there are elements in the domain of interpretation which can serve to represent finite sets, it is possible to simulate this weak second order quantification by first order quantifiers. Any infinite set of finitely generated elements will serve to represent finite sets, so, aside from the pathological case of (essentially) finite domains, we can extend the theorem to nondeterministic effective flowcharts even with array assignments.

Namely, let \( \Sigma \) be some finite set of function symbols. A state is \( n, \Sigma \)-infinite iff there are \( n \) elements of the domain of the state such that the set of elements generated by applying the functions (which are the interpretations in the state of the symbols) in \( \Sigma \) to these \( n \) elements is infinite.

**Theorem 4:** For any \( n > 0 \) and finite set \( \Sigma \) of function symbols, there is an effective procedure to translate any formula \( p \) of the logics 9.-11. below, into a formula \( p' \) of \( \mathcal{L}_{re} \) such that for every \( n, \Sigma \)-infinite state \( s \),

\[
 s \models p \iff s \models p'.
\]

9. \( DL^+ \) of nondeterministic effective flowcharts,

10. Logic of nondeterministic effective definitional schemes (with array assignments),

11. AL of nondeterministic effective flowcharts without the iteration quantifier \( \sqcap \).

It remains an interesting open question whether the hypothesis of \( n, \Sigma \)-infinity can be eliminated from Theorem 4. Whether the iteration quantifier \( \sqcap \) makes a difference in the presence of nondeterministic programs is also open, but appears to be of technical interest only.

In the next section we present the main definitions among the logics 1.-11., and prove Theorems 3 and 4.
2 Definitions and Proofs

All of the logics 1.-11. are subsets of the following class $L_{\text{univ}}$ of formulas which is obtained by combining the features of all the languages.

Definition 5: $L_{\text{univ}}$ is defined inductively as follows:

(a). Any atomic formula of predicate calculus with equality is a formula of $L_{\text{univ}}$.

(b). If $\alpha$ is an effective flowchart, then $LOOPS_\alpha$ is a formula of $L_{\text{univ}}$.

(c). If $p, q$ are formulas of $L_{\text{univ}}$, then so are $\neg p, p \land q, p \lor q, \exists x[p], \forall x[p]$.

(d). If $P_1, P_2, \ldots$ is an r.e. sequence of formulas of $L_{\text{univ}}$, then so are $\vee\{P_i \mid i \geq 1\}$ and $\land\{P_i \mid i \geq 1\}$.

(e). If $\alpha$ is an effective flowchart and $p$ is a formula of $L_{\text{univ}}$, then so are $<\alpha>p$ and $[\alpha]p$.

(f). If $\alpha$ is an effective flowchart and $p$ is a formula of $L_{\text{univ}}$, then so are $(\cap \alpha)p$ and $(\cup \alpha)p$.

Whether a state $s$ satisfies a formula $p$ of $L_{\text{univ}}$, denoted $s \models p$, is defined in the usual way for $p$ of the form (a), (c), or (d) above.

For case (b), $s \models LOOPS_\alpha$ iff $\alpha$ loops from state $s$.

For case (e), $s \models <\alpha>p$ iff $t \models p$ for some state $t$ such that $(s, t) \in R_\alpha$; $s \models [\alpha]p$ iff $t \models p$ for all states $t$ such that $(s, t) \in R_\alpha$.

Case (f) covers the iteration quantifiers of AL. $s \models (\cup \alpha)p$ iff $s \models <\alpha^*>p$, where $\alpha^*$ is an effective flowchart such that $R_{\alpha^*}$ is the reflexive transitive closure of $R_\alpha$. $s \models (\cap \alpha)p$ iff $s \models <\alpha^n>p$ for all $n \geq 0$, where $\alpha^n$ is an effective flowchart such that $R_{\alpha^n} =$ the relational composition of $R_\alpha$ with itself $n$ times.

This defines the semantics of $L_{\text{univ}}$. 
L_{re} is easily embeddable in all of the logics of Theorem 3, and all are obviously embeddable into one of DDL or DL^{+-w/o-array}, so we give precise definitions and proofs only for these latter two logics.

**Definition 6:** DDL is the class of formulas defined by rules (a,c,e) of Definition 5 such that the flowcharts \( \alpha \) of rule (e) are deterministic. DL^{+-w/o-array} is the class of formulas defined by rules (a,b,c,e) such that the flowcharts \( \alpha \) of rule (e) do not contain array assignments.

To prove Theorem 3, we describe translations between L_{re} and DDL, and between L_{re} and DL^{+-w/o-array}.

The translation from L_{re} actually takes formulas of L_{re} into the intersection of DDL and DL^{+-w/o-array}. It is obtained trivially from the observation that the atomic formula \( \vee \{P_i \mid i \geq 1 \} \) of L_{re} is equivalent to \( \langle \alpha \rangle \text{true} \) where \( \alpha \) is the effective flowchart

```plaintext
if P_1 then x := x else
if P_2 then x := x else
if P_3 then x := x else....
```

The translation from DDL to L_{re} is based on

**Lemma 7:** The following formulas are valid for any flowchart \( \alpha \) and formula \( p \) of L_{univ}:
1. \( \langle \alpha \rangle (p \vee q) \equiv (\langle \alpha \rangle p \vee \langle \alpha \rangle q) \).
2. \( \langle \alpha \rangle \exists x[p] \equiv \exists z[\langle \alpha \rangle (p[z/x])] \), where \( z \) does not occur in \( \alpha \) or \( p \), and \( p[z/x] \) is the result of substituting \( z \) for \( x \) in \( p \).

In addition, the following formula is valid for any deterministic flowchart \( \alpha \) and formula \( p \) of L_{univ}:
3. \( \langle \alpha \rangle \neg p \equiv (\langle \alpha \rangle \text{true} \land \neg \langle \alpha \rangle p) \).

The equivalences of Lemma 7 allow one to "move the \( \Diamond \)'s in" thereby converting an arbitrary formula of DDL into an equivalent formula built solely by first order constructs, i.e., the rules of Definition 5.(c), starting from formulas of the form \( \langle \beta_1 \rangle ... \langle \beta_n \rangle P \) where \( P \) is an atomic formula of predicate calculus. But a formula of the form \( \langle \beta_1 \rangle ... \langle \beta_n \rangle P \) is
equivalent to an r.e. disjunction of formulas $<\alpha_i>P$ where $\alpha_i$ ranges over the terminating instruction sequences of the program $\beta_1;\ldots;\beta_n$. Each formula $<\alpha_i>P$, where $\alpha_i$ is a finite sequence of assignments and atomic tests and $P$ is quantifier free, is equivalent to a quantifier free formula of predicate calculus, cf. PRATT 76, MEYER and PARIKH 80. In this way DDL translates into $L_{re}$.

The translation from $DL^+\text{-w/o-array}$ into $L_{re}$ proceeds by induction on the definition of $DL^+$. The only interesting case in the basis of the induction is for formulas of the form $LOOPS_{\alpha}$. These are obviously equivalent to the r.e. conjunction of the quantifier-free first order formulas which assert that a terminating instruction sequence in $\alpha$ is not executable.

The essential step in the inductive definition of the translation is $<\cdot>$-elimination. Let $\alpha$ be a nondeterministic effective flowchart without array assignments and let $p$ be a formula of $DL^+\text{-w/o-array}$. By induction, we may assume there is a formula $q$ of $L_{re}$ equivalent to $p$. Let $x_0,\ldots,x_{n-1}$ be all the variables occurring in flowchart $\alpha$. It is easy to define an r.e. set of quantifier-free first order formulas $\{P_i| i \geq 0\}$ and an r.e. set of terms $\{\tau_{ij}| i \geq 0, j < n\}$ such that for all $j < n$ and states $s$, $s \models P_j$ iff it is possible for $\alpha$, started in state $s$, to terminate with the terminal value of $x_j$ equal to the value of $\tau_{ij}$ in state $s$.

Let $y_0,\ldots,y_{n-1}$ be new variables which occur neither in $\alpha$ nor in $q$. The reader can easily check that

$$3y_0,3y_{n-1}[\vee_{i\{P_i \land (\wedge_{j \leq n} y_j = \tau_{ij})\} \land q[y_0,\ldots,y_{n-1}/x_0,\ldots,x_{n-1}]]$$

is equivalent in all states to $<\alpha>P$.

We remark that introducing quantifiers in formula (1), or indeed any such formula which accomplishes $<\cdot>$-elimination, is unavoidable. This follows from the fact that the quantifier-free fragment of Deterministic $DL^+\text{-w/o-array}$, which is equivalent to quantifier-free $L_{re}$, is strictly weaker than the quantifier-free fragment of $DL^+\text{-w/o-array}$, (cf. MEYER and WINKLMANN 80).
This completes the proof of Theorem 3.

In proving Theorem 4, we note that all of the logics 9.-11. are no more expressive than \( \text{DL}^+ \). We therefore only describe the translation of \( \text{DL}^+ \) into \( L_{\text{re}} \).

As in the proof of Theorem 3, the translation is given inductively. The only interesting case is \( \triangleright \text{-elimination} \).

Let \( p \) be a formula of \( \text{DL}^+ \) and let \( \alpha \) be a nondeterministic effective flowchart. By induction, let \( q \) be an \( L_{\text{re}} \) formula equivalent over all \( n, \Sigma \)-infinite states to \( p \). According to Lemma 1 we can find a nondeterministic effective definitional scheme which defines the same input-output relation as \( \alpha \). This effective definitional scheme is an infinite parallel \textbf{OR} of finite deterministic programs \( \alpha_i \) of the form

\[
\text{if } P_i \text{ then ASSIGN}_i \text{ else ABORT fi}
\]

where ASSIGN\(_i\) is a finite sequence of assignments.

Obviously \( \langle \alpha \rangle p \) is equivalent to the \( L_{\text{univ}} \) formula

\[
\forall j \langle \alpha_j \rangle q.
\]  

(2)

It is not hard to show that any formula \( \langle \alpha_j \rangle q \) is equivalent to a formula of \( L_{\text{re}} \), but this still leaves the difficulty that (2) is an infinite disjunction of \( L_{\text{re}} \), not first order, formulas, and \( L_{\text{re}} \) is not closed under infinite disjunctions. We could eliminate this difficulty if the integer variable \( i \) in \( \langle \alpha_j \rangle q \) could somehow be taken as a variable of \( \text{DL} \), for then the infinite disjunction over \( i \) in (2) could simply be replaced by an existential quantification of \( i \). With the aid of the hypothesis of \( n, \Sigma \)-infinity, we will accomplish this as follows.

Each formula \( \langle \alpha_j \rangle q \) can be transformed using the equivalences of Lemma 7 so that all the occurrences of \( \langle \alpha_i \rangle \) appear in the context

\[
\langle \alpha_j \rangle \forall_m G_m
\]

(3)

where the \( G_m \) are quantifier-free formulas of predicate calculus. Let the formula obtained
in this way be denoted \( q_i \), so that formula (2) is equivalent to \( \lor_i q_i \). (The transformation is uniform in \( i \), so the same set of disjunctions \( \lor_m G_m \) occur in each \( q_i \)) in order to eliminate the outermost disjunction in (2) we use the assumption of \( n, \Sigma \)-infinity of states.

Let \( y = y_0, \ldots, y_n-1 \), and \( z \) be \( n+1 \) individual variables which occur neither in \( \alpha \) nor in \( q_i \), and choose some effective enumeration \( \tau_1(y), \tau_2(y), \ldots \) of all the terms over \( y \cup \text{signature}(\alpha) \cup \text{variables}(\alpha) \), i.e., the terms with function symbols from \( \alpha \) whose only variables are from \( y \).

For \( k, i \geq 1 \), let \( D_{k,i}(y,z) \) be a quantifier-free formula of predicate calculus which expresses the following property: "\( z \) is the value of the \( k \)-th term (in the above enumeration), there are exactly \( i \) distinct values among those first \( k \) terms, and \( k \) is the least integer with the above two properties".

Let \( q' \) be a formula obtained from \( q_i \) by replacing every subformula of the form (3) by the r.e. disjunction

\[
\lor \{D_{k,j}(y,z) \land G_{mj} | j, k, m \geq 1\}
\]

where \( G_{mj} \) is a quantifier free first order formula equivalent to \( \langle\alpha_j \rangle G_m \). Note that by the uniformity in \( i \) of the definition of \( q_i \), it follows that the same \( q' \) is obtained for all \( i \).

In \( q' \) we have apparently eliminated the index \( i \), but it will be coded in the values of variables \( y \) and \( z \). This coding is possible because, by definition of \( D_{k,j} \), for every state \( s \) there is at most one pair of integers \( k, j \geq 1 \) such that \( s \models D_{k,j} \). Moreover, for every \( n, \Sigma \)-infinite state \( s \) and for arbitrary \( k, j \geq 1 \), \( s \models \exists y \exists z[D_{k,j}(y,z)] \). Thus in \( n, \Sigma \)-infinite states we can code any pair of integers by using the formulas \( D_{k,j} \).

We use the above observation to code the value of index \( i \). Let \( q'' \) be the formula

\[
\exists y \exists z[\lor \{P_i \land D_{k,i}(y,z) | k, i \geq 1\} \land q']
\]

where \( P_i \) is the test portion of \( \alpha_i \).
We claim that for every $n, \Sigma$-infinite state $s$,
\[
\models \forall i < \alpha_i \models q \equiv q'.
\] (4)

In order to prove the claim (4), let us assume that $s \models < \alpha_i \models q$ for a certain $i \geq 1$. Let $a = a_0, \ldots, a_{n-1}$ be the generators of an infinite substructure in $s$, let $b$ be the $i$-th distinct value in the sequence $\tau_1(a), \tau_2(a), \ldots$ and let $k \geq 1$ be the least integer such that $b = \tau_k(a)$. Let $s_i$ be the state in which $y$ has the value $a$, $z$ has the value $b$, and all other symbols have the same interpretation as in $s$. We have $s_i \models P_1 \land D_{k,i}(y,z)$ because $s_i \models P_1$ and $y,z$ do not occur in $P_1$.

In order to see that $s_i \models q'$, it is enough to observe that for any r.e. set of formulas $\{G_m\}$ \(m \geq 1\),
\[
s_i \models \forall \{D_{k,j}(y,z) \land G_{mj}\} \land j,k,m \geq 1 \text{ iff } s \models < \alpha_i \forall \forall G_m.
\]

In this way we have proved $s \models q'$. The other half of the equivalence (4) is proved similarly.

3 Conclusion

Having reduced essentially all the various programming logics to the $L_{re}$ fragment of infinitary logic, it is easy to deduce a body of model theoretic and undecidability results about programming logic from known results for infinitary logic. Moreover, the reduction to $L_{re}$ is sufficiently straightforward that various infinitary proof theoretic results can also be carried over directly to programming logic.

We interpret these results as evidence that no very new model theoretic or recursion theoretic issues arise from logics of programs on first order structures.

Nevertheless, we believe that the problem of developing formal systems for reasoning about programs offers significant challenges in at least two directions. First, to be true to the purpose for which high level programming languages were originally developed and continue to be developed -- namely for economy and ease in the expression of algorithms --
it is important to develop proof methods for dealing with high level programs as textual objects. This has in fact been the focus of the bulk of the literature on program correctness, although many of the complex features of modern programming languages have yet to be adequately addressed. (In our treatment we assumed in effect that the high level programs had already been transformed into effective flowcharts, and thereby we avoided the challenge of developing a proof theory.) A second challenge involves programs operating on higher-type domains which are often assumed to satisfy "domain equations" which appear inconsistent with standard set theory. Development of appropriate logics for reasoning about such domains has just begun, cf. SCOTT 80, and seems an intriguing subject for further research.

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