LIGHT TRAPS

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1. Introduction

In the February 1992 issue of the *American Mathematical Monthly*, J. E. Connett [1] asked whether it is possible to construct a 'light trap': a reflective-sided container with the property that a beam of light, shone into it from an appropriate direction, would be reflected inside it over and over again and never escape. Connett suggests that such a trap might be of value as a device to store light rays; however, the market for escape-proof golf holes might be even more lucrative!

In order to make the problem interesting, we should insist that the container be of finite size, and that the walls of the container be smooth everywhere, or at least everywhere where the beam of light hits. It is not sufficient to focus the beam of light to a spot on the wall of the container where the container fails to be smooth, and hence where the reflection of the beam is not well-defined. Rather, the beam must be reflected in such a way that it can keep traveling forever without leaving the container.

2. Examples of light traps

Our first, simplest, example of such a light trap consists of two confocal parabolic mirrors, one reflecting on the inside and one on the outside (Figure 1), joined by an arbitrary smooth arc. If a beam of light crosses the segment ab parallel to the y-axis, it will be reflected at a point p_1, towards the focus f. The reflected beam will strike the inner mirror at a point q_1, and be reflected to be once again parallel to the y-axis. Continuing in this manner we obtain a sequence of reflection points $(p_1, q_1, p_2, q_2, \ldots, p_i, q_i, \ldots)$.

It is clear that, for any line through f meeting the two parabolic arcs in points p_i and q_i as shown,

$$0 < \frac{|f - q_i|}{|f - p_i|} < 1.$$

As the arcs are compact, we can find α, β such that

$$0 < \alpha < \frac{|f - q_i|}{|f - p_i|} < \beta < 1.$$

It is clear that, if the original beam enters along the line $x = x_0$, the x-coordinate x_i of q_i (and of p_{i+1}) satisfies

$$0 < \alpha^i x_0 < x_i < \beta^i x_0$$

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and thus as $i \to \infty$ the successive reflected rays approach the y-axis but never reach or cross it. From this, it follows that the beam can never escape.

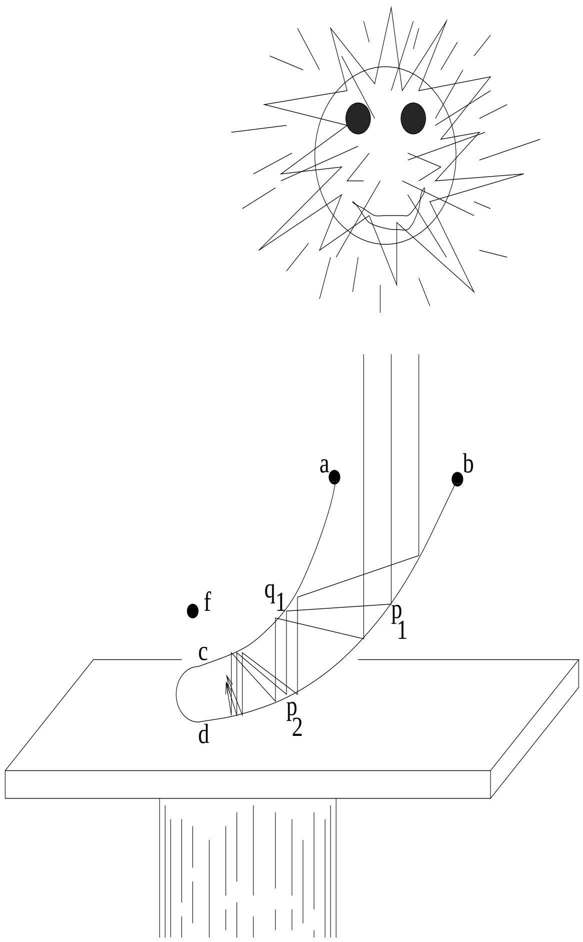

Figure 1

This construction may clearly be extended into three or more dimensions, creating, for example, a three-dimensional light trap between two confocal paraboloids. disappointingly, the three dimensional theory of light traps does not seem to have any new surprises for us; it seems that once we know about the properties of light traps in the plane, the higher-dimensional theories are simple generalizations.

The light trap of Figure 1 has the property that nowhere along the beam of light is there a focal point at which the light is infinitely bright. However, this is somewhat misleading; for while the light is not focused anywhere in the conventional sense, any beam which crosses ab parallel to the y-axis passes infinitely often through any open neighborhood of any point on the chord cd. So, while no point on cd is illuminated, any neighborhood of such a point is, in a sense, illuminated infinitely brightly!

Can we create a *convex* light trap? Yes; Figure 2 shows an example of such a curve, obtained by flattening a short section cd of an ellipse into a parabola.

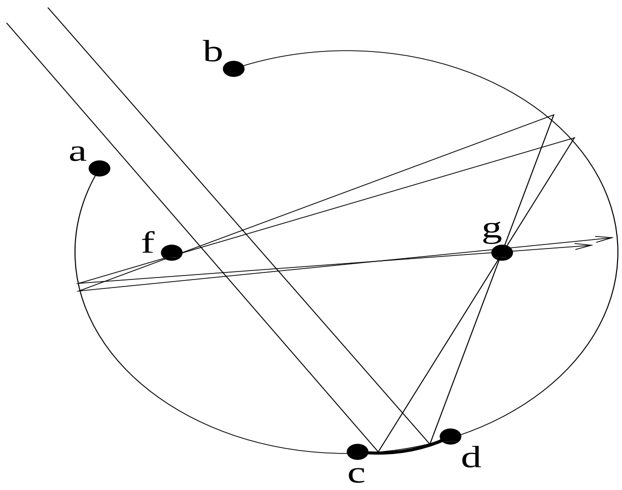

Figure 2

To see that this actually works, note that if x is on the boundary of the ellipse to the left of the right focus g, and if the ray from x through the right focus g meets the ellipse again at a point y, then y is to the right of g, and

$$\frac{|x-g|}{|y-g|} > \frac{1+\epsilon^2}{1-\epsilon^2} > 1,$$

where ϵ is the eccentricity of the ellipse. (This property of an ellipse is well-known; see, for instance, [4], page 239-241). Thus, each successive reflection will be strictly closer to the x-axis than the previous one, and we may create a window on one side of the axis, and modify the shape of a small region on the other, without affecting the region in which any later reflections will take place.

We remove a section ab of the ellipse, as the window through which light will enter, and select cd to be a region to the left of the focus g. We now flatten the arc cd from the elliptical arc (which would focus a *divergent* pencil of rays from f to g) to a parabolic arc which will focus a *parallel* beam coming through the window ab onto g. This is clearly flatter than the original elliptical arc, but still convex. Thus, the resulting light trap is convex (similar constructions to this one have been discovered independently, see [3], [2]).

Can we design a light trap that will trap light from more than one direction? Such a trap would be necessary if we wished to trap diffuse light, or to have some margin for error in the orientation of the trap. Clearly, simple modifications of the designs above will allow us to trap a pencil of rays coming from all directions but focussed through one point; for instance, consider an ellipse with a window, which will trap a pencil of beams focussed through f and striking the boundary to the left of f. We can also trap pencils of rays from a finite, or even countable, number of directions; imagine the arc cd in Figure 2 broken up into an infinite sequence of little parabolic mirrors, of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, each focusing light from a different direction onto the focus g (Figure 3).

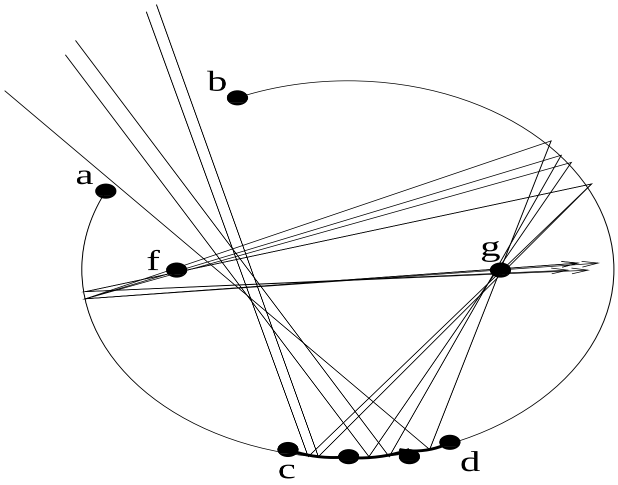

Figure 3

Unfortunately, in the following section we show that we cannot hope to do much more than that:

3. Trapping light from arbitrary directions

Before we can state a theorem we must give a precise definition of what we will consider a 'light trap':

Definition 1. *A light trap C is a subset of $\mathbb{R}^n$ with smooth boundary and finite n dimensional volume V.*

A *light ray* will be thought of as consisting of a set of points in $\mathbb{R}^n$. A point along the ray is defined by its position and the direction in which the ray is traveling at that point:

Definition 2. *A $2n-1$ tuple in $\mathbb{R}^n \times S^{n-1}$ consisting of the position of a point and the direction it is traveling in will be called a path point.*

The *phase space* is the set of all possible path points. Because the coordinates of a point in the phase space contain enough data to determine the future behavior of the 'point of light', each point lies on a 'trajectory' which corresponds to its past and future history. The following elementary observation is important:

Note 1. If two 'points of light' coincide in phase space, they are not only in the same place but going in the same direction; thus they have been, and will remain, coincident forever.

We are interested in studying the behavior of light as it bounces around in the trap. Let the initial set of path points in the trap be denoted by $S(0)$. Let t be the variable corresponding to time and let $S(t)$ be the set of path points obtained from $S(0)$ after the light has reflected for a period of time t inside the trap. Assume, also, that each point in $S(0)$ has been inside the trap at most time ϵ.

Let dx be the uniform distribution on $\mathbb{R}^n$ and let $d\omega$ be the uniform distribution on S^{n-1}. Take the measure on $\mathbb{R}^n \times S^{n-1}$ to be the product measure $dx * d\omega$. If the measure of S^{n-1} is Ω then the measure of the trap in $\mathbb{R}^n \times S^{n-1}$ is ΩV.

Theorem 1. *Let C be a light trap with finite n dimensional volume V. If*

$$\int_{S(0)} dx * d\omega > 0$$

then some light will escape from the trap.

The quantity $\int_{S(0)} dx * d\omega$ measures how much phase space is 'occupied' by the light at any instant. The theorem states that if the initial light has nonzero measure in the phase space, it cannot be trapped.

Proof of theorem: We begin with the following crucial lemma:

Lemma 1.

$$\int_{S(t)} dx * d\omega$$

is invariant with respect to time.

Proof of lemma: Observe that the measure $dx * d\omega$ is invariant under translation and rotation. Equivalently, the quantity above is preserved if the light does not reflect, or reflects off a flat mirror. By approximating the mirror with polyhedral mirrors we can show that any reflection is measure preserving.

Let $S'(t)$ be a set of path points with the property that $S'(t)$ reflects off of C between time t and $t + \delta$ only once (where δ has been chosen to be sufficiently small). Notice that $S'(t + \delta) = F(S'(t))$ for some smooth function F. Let

$$E = \int_{S'(t)} dx * d\omega.$$

Consider a polygonal approximation to the mirror both in position and orientation.

Partition $S'(t)$ into sets $S'_k(t)$ such that $S'_k(t)$ is a set of path points reflecting off of only one face of the polygonal approximation. Let T be the set of path points in $S'(t)$ that go into a corner (this light is lost).

Thus,

$$S'(t) = T \cup \bigcup_k S'_k(t)$$

where T has measure 0. Notice that for each k, there exists a smooth function f_k such that

$$S'_k(t + \delta) = f_k(S'_k(t)).$$

Given any integer n, pick a polygonal approximation good enough so that

$$|J_{f_k}(\overline{x}) - J_{F|_{S'_k(t)}}(\overline{x})| < \frac{1}{nE} \ \forall k, \forall \overline{x} \in S'_k(t).$$

Here J denotes the Jacobian and $F|_{S'_k(t)}$ is F restricted to $S'_k(t)$.

Observe that

$$|\sum_k \int_{S'_k(t)} J_{F|_{S'_k(t)}}(\overline{x})dx * d\omega - \sum_k \int_{S'_k(t)} J_{f_k}(\overline{x})dx * d\omega|$$

$$\leq \sum_k \int_{S'_k(t)} |J_{f_k}(\overline{x}) - J_{F|_{S'_k(t)}}(\overline{x})|dx * d\omega$$

$$\leq \sum_k \int_{S'_k(t)} \frac{1}{nE} dx * d\omega = \int_{S'(t)} \frac{1}{nE} dx * d\omega = \frac{1}{n}.$$

Thus, the error introduced by approximating the mirror is arbitrarily small, hence the lemma. □

Lemma 2. *Let* $t_k = k\epsilon$ *and let* m, n *be positive integers with* $n \neq m$. *The sets* $S(t_n)$ *and* $S(t_m)$ *satisfy*

$$S(t_n) \cap S(t_m) = \varnothing.$$

Proof of lemma: The proof is by induction. Clearly if $n = 0$ then

$$S(t_0) \cap S(t_m) = \varnothing$$

for $m \geq 1$. This is because we are assuming that the initial light came from outside the trap.

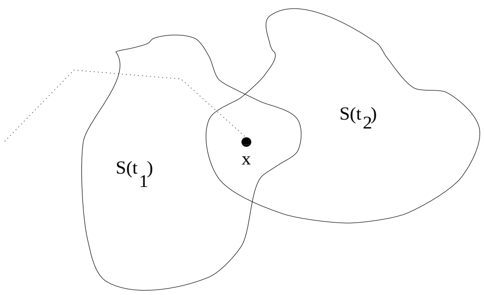

Figure 4

Similarly,

$$S(t_n) \cap S(t_m) = \varnothing$$

when $m \geq n$.

Suppose $x \in S(t_n) \cap S(t_m)$ and $n \geq 1$. It must be the that $x \in S(t_{n-1}) \cap S(t_{m-1})$, but by induction the two sets are disjoint. □

By Lemma 1, $S(0), S(\epsilon), S(2\epsilon), \ldots$ all have the same nonzero measure and by Lemma 2 they are disjoint sets. The region of phase space corresponding to the interior of the trap has finite measure ΩV and thus the sets $S(0), S(\epsilon), S(2\epsilon), \ldots$ cannot all lie inside the trap, and some light must eventually escape. □

References

[1] J. E. Connett, "Trapped Reflections?" *American Mathematical Monthly*, **99** (1992), p 179-179.

[2] M. Frantz, "A Focusing Property of the Ellipse," *American Mathematical Monthly*, **101** (1994), p 250-258.

[3] R. Peirone, "Reflections Can be Trapped," *American Mathematical Monthly*, **101** (1994), p 259-260.

[4] H. Steinhaus, *Mathematical Snapshots*, New York 1969.

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