Fixed-parameter algorithms for minor-closed graphs (of locally bounded treewidth) *

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Abstract. Frick and Grohe [7] showed that for each property ϕ that is definable in first-order logic, and for each class of minor-closed graphs of locally bounded treewidth, there is an $O(n^{1+\epsilon})$-time algorithm deciding whether a given graph has property ϕ. In this paper, we extend this result for fixed-parameter algorithms and show that any *minor-closed [contraction-closed] bidimensional parameter* which can be computed in polynomial time on graphs of bounded treewidth is also fixed-parameter tractable on general minor-closed graphs [minor-closed class of graphs of locally bounded treewidth]. These parameters include many domination and covering parameters such as vertex cover, feedback vertex set, dominating set, and clique-transversal set. Our algorithm is very simple and its running time is explicit (in contrast to the work of [7]). Along the way, we obtain interesting combinatorial bounds between the aforementioned parameters and the treewidth of the graphs.

1 Introduction

Developing fast algorithms for NP-hard problems is an important issue. Downey and Fellows [5] introduced *fixed-parameter tractability* as an approach to cope with NP-hardness. For many NP-complete problems, the inherent combinatorial explosion can be attributed to a certain aspect of the problem, a *parameter*. The parameter is often an integer and small in practice. The running times of simple algorithms may be exponential in the parameter but polynomial in the rest of the problem size. So far, many authors have presented fixed-parameter algorithms for many NP-hard problems on general or special class of graphs; see [5] for a survey.

Many problems which are NP-complete or fixed-parameter intractable on general graphs can be solved or approximated on graphs of bounded treewidth or on planar graphs. Eppstein [6] was the first who introduced the concept of *locally bounded treewidth* which effectively extends good algorithmic properties of planar graphs and graphs of bounded treewidth to a more general class of graphs.

Courcelle [2] proved a meta-theorem on graphs of bounded treewidth; he showed that, if ϕ is a property of graphs that is definable in monadic second-order logic, then ϕ can be decided in linear time on graphs of bounded treewidth. Frick and Grohe [7] extended this result to graphs of locally bounded treewidth; they showed that, for each property ϕ that is definable in first-order logic and for each minor-closed class of graphs of locally bounded treewidth, there is an $O(n^{1+\epsilon})$-time algorithm deciding whether a given graph has property ϕ. However, the running times of both of these results have huge hidden constants which are not even mentioned explicitly. In this paper, we extend this result for fixed-parameter algorithms and show that any *minor-closed [contraction-closed] bidimensional property* that is solvable in polynomial time on graphs of bounded treewidth is also fixed-parameter tractable on general minor-closed graphs [minor-closed class of graphs of locally bounded treewidth]. In contrast to the work of Frick and Grohe [7], the running time of our algorithm is explicit and applies simple combinatorial bounds relating the aforementioned parameters and the treewidth of the graphs.

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2 Definitions

Contracting an edge $e = \{u, v\}$ in an (undirected) graph G is the operation of replacing both u and v by a single vertex w whose neighbors are all vertices that were neighbors of u or v, except u and v themselves. A graph G is a *minor* of a graph H if H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{C}$ is a *minor-closed* class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor-free* if $H \notin \mathcal{C}$.

A *parameter* P is any function mapping graphs to nonnegative integers. The *parameterized problem* associated with P asks, for some fixed k, whether $P(G) \leq k$ for a given graph G. A parameter P is called *minor-closed [contraction-closed] $f(r)$-bidimensional*[1] if (I) contracting or deleting [contracting] an edge in a graph G cannot increase $P(G)$, and (II) there exists a function f, such that for the $(r \times r)$-grid [any $(r \times r)$-augmented grid], $P(R) \geq f(r)$. Here an *$(r \times r)$-augmented grid* is an *$(r \times r)$-grid* (intersection graph of r rows and r columns) with some extra edges such that each vertex is attached to only a bounded number of non-boundary vertices. We assume that $f(r)$ is monotone and invertible for $r \geq 0$. One can easily observe that many parameters such as dominating set, vertex cover, feedback vertex set, clique transversal set and, edge dominating set are minor- or contraction-closed $\Theta(r^2)$-bidimensional parameters.

Representation of a graph as a tree with a *tree decomposition* plays an important role in design of algorithms. A *tree decomposition* of a graph $G = (V, E)$, denoted by $TD(G)$, is a pair (χ, T) in which $T = (I, F)$ is a tree and $\chi = \{\chi_i \mid i \in I\}$ is a family of subsets of $V(G)$ such that: (1) $\bigcup_{i \in I} \chi_i = V$; (2) for each edge $e = \{u, v\} \in E$, there exists an $i \in I$ such that both u and v belong to χ_i; and (3) for all $v \in V$, the set of nodes $\{i \in I \mid v \in \chi_i\}$ forms a connected subtree of T. The maximum size of χ_i's in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G ($\mathrm{tw}(G)$) is the minimum width over all possible tree decompositions of G. A graph has *bounded local treewidth* (or *locally bounded treewidth*) if the treewidth of the subgraph induced by the set of vertices at distance at most r from any vertex is bounded above by some function $f(r)$ independent of n. Eppstein [6] showed that minor-closed graphs of bounded local treewidth are precisely apex-minor-free graphs, where an *apex graph* has a vertex whose removal leaves a planar graph. We use this characterization in the rest of the paper.

3 Combinatorial Bounds

In this section, we show minor-closed graphs of locally bounded treewidth (or apex-minor-free graphs according to Eppstein's characterization) are the most general class of graphs for which there exist bounds between their contraction-closed bidimensional parameters and their treewidth. Before stating this result in Theorem 2, we present a theorem on general minor-closed classes of graphs for minor-closed bidimensional parameters, whose intuition plays an important role in Theorem 2.

Theorem 1. *If a minor-closed $f(r)$-bidimensional parameter P on an H-minor-free graph G has value at most k, then* $\mathrm{tw}(G) \leq 2^{O(f^{-1}(k) \log f^{-1}(k))}$.

Proof. By results of Robertson and Seymour [10] and Diestel et al. [4], because G is H-minor-free, we know that if $\mathrm{tw}(G) \geq r^{4|V(H)|^2(r+2)}$, then G has an $(r \times r)$-grid as a minor. Because P is minor-closed $f(r)$-bidimensional, $k \geq f(r)$ and thus we obtain the desired bound. □

Theorem 2. *If a contraction-closed $f(r)$-bidimensional parameter P on a graph G excluding an apex graph H as a minor has value at most k, then* $\mathrm{tw}(G) \leq 2^{O(f^{-1}(k) \log f^{-1}(k))}$.

Proof. As mentioned before, because G is H-minor-free, if $\mathrm{tw}(G) \geq r^{4|V(H)|^2(r+2)}$, then G has an $(r \times r)$-grid as a minor. Suppose this is the case for a value r to be determined later. Now, during

[1] A closely related notion of bidimesional parameters is introduced by Demaine, Fomin, Hajiaghayi, and Thilikos [3].

the course of taking a minor to obtain such a grid, we apply only contractions and ignore deletions. Call the new graph R, which is in fact a grid with some extra edges.

Let x be a vertex in R and let $N(x)$ be the neighbors of x in R (i.e., we consider the neighbors of x after the contractions). We show that the number of neighbors of x in the central $((r-2N_2)\times(r-2N_2))$-grid of R is bounded by some constant depending only on the excluded apex graph H, where N_2 is also a constant depending only on H. To this end, we "lift" x from the grid—not removing it from the graph per se, but marking it as "outside the grid." Then we contract the remainder of x's column and the two adjacent columns (if they exist) into a single column. Similarly we contract the remainder of x's row with the adjacent rows. The new graph R' is a grid with extra edges and an extra vertex x.

Now consider neighbors $N'(x)$ of x in R'. By the theory of Robertson and Seymour (see [8, Section 9] or [9, Theorems 4.4 and 4.5]), if we have h vertices on an $(r\times r)$-grid such that each pair has distance at least some constant N_2 and each vertex has distance at least N_2 from the boundary of the outer region, then by taking a minor of the $(r\times r)$-grid, we can construct any planar graph whose vertex set is precisely these h vertices. (In fact, Robertson and Seymour consider the distance metric of the tangle of the graph, but for grids, this distance is proportional to the normal graph distance, except for vertices near the boundary of the grid that we exclude above.) For our case, because G is H-minor-free and H is an apex graph, we let $h=|V(H)|-1$. Therefore by this theorem, for each vertex x, there is a set $C_x\subseteq N'(x)$ of at most $h=|V(H)|-1$ vertices (called *centers*) such that the distance in R' between any pair of vertices in C_x is at least N_2, and every vertex of $N'(x)$ in the central $((r-2N_2)\times(r-2N_2))$-grid of R' has distance at most N_2 from one of the vertices of C_x. In other words, all vertices of $N'(x)$ in the central grid can be covered by at most h disks of radius N_2. Thus, the degree of vertex x in the central grid of R' is at most $4hN_2^2$. Because our contraction scheme from R to R' has the property that at most eight vertices in R are contracted into one vertex in R', the degree of each vertex in the central $(r-2N_2\times r-2N_2)$-grid of R is at most $8hN_2^2$.

Now, suppose in graph R, we further contract all $2N_2$ boundary rows and $2N_2$ boundary columns into two boundary rows and two boundary columns (one on each side) and call the new graph R''. The degree of each vertex of R'' to the vertices that are not on the outer region (boundary) is at most $2N_2 8h_1N_2^2$, which is a constant depending only on H. Here the factor $2N_2$ is for the boundary vertices which are obtained by contraction of at most $2N_2$ vertices. Because the parameter P is contraction-closed, its value on graph R'' is at most k, its original value on graph G. Since the parameter is $f(r)$-bidimensional, its value on graph R'' is at least $f(r-2N_2)$. Thus $k\geq f(\Theta(r))$, i.e., $f^{-1}(k)\geq\Theta(r)$, so the treewidth of the graph is at most $2^{O(f^{-1}(k)\log f^{-1}(k))}$ as desired. □

One can easily observe that Theorem 2 is tight in the sense that, for example, dominating set is contraction-closed $\Theta(r^2)$-bidimensional, and if G is the apex graph formed by connecting a vertex v to all vertices of an $(r\times r)$-grid, then G has dominating set of size 1 and treewidth $r+1$ $(r=O(\sqrt{n}))$. Thus there is no relation between the treewidth and the size of dominating sets for apex graphs.

4 Algorithmic Applications

Using Theorem 2, we obtain the following algorithmic result.

Theorem 3. *Let P be a parameter which can be decided on graphs of treewidth at most w in $g(w)n^{O(1)}$ time. Now, if P is minor-closed $f(r)$-bidimensional parameter and G belongs to a minor-closed family of graphs, or P is contraction-closed $f(r)$-bidimensional parameter and G is belongs to a minor-closed family of graphs of locally bounded treewidth, then we can decide P on G in $g(2^{O(f^{-1}(k)\log f^{-1}(k))})n^{O(1)}+2^{2^{O(f^{-1}(k)\log f^{-1}(k))}}n^{3+\epsilon}$ time.*

Proof. First, using Theorems 1 and 2, we know that if P on G has value at most k, then the treewidth of G is at most $2^{O(f^{-1}(k)\log f^{-1}(k))}$. Now, using a result of Amir [1], we can construct a tree decomposition of width $(3+2/3)2^{O(f^{-1}(k)\log f^{-1}(k))}$ in time $O(2^{3.6982^{O(f^{-1}(k)\log f^{-1}(k))}}n^{3+\epsilon})$. (If treewidth is greater than the aforementioned value, the algorithm rejects the graph.) Now, having the tree decompostion, we solve the problem in $g(2^{O(f^{-1}(k)\log f^{-1}(k))})n^{O(1)}$ time. $\square$

We note that, in Theorem 3, the hidden constants are small, and the large constants are made explicit, unlike [7]. Using Theorem 3, we can obtain fixed-parameter algorithms with running time $2^{2^{O(\sqrt{k}\log k)}}n^{O(1)}$ for many problems such as k-dominating set, k-vertex cover, k-feedback vertex set, k-clique transversal set, and k-edge dominating set.

References

1. Eyal Amir. Efficient approximation for triangulation of minimum treewidth. In *Uncertainty in Artificial Intelligence: Proceedings of the Seventeenth Conference (UAI-2001)*, pages 7–15, San Francisco, CA, 2001. Morgan Kaufmann Publishers.
2. Bruno Courcelle. Graph rewriting: an algebraic and logic approach. In *Handbook of theoretical computer science, Vol. B*, pages 193–242. Elsevier, Amsterdam, 1990.
3. Erik D. Demaine, Fedor V. Fomin, Mohammadtaghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algoirthms on graphs of bounded genus and H-minor-free graphs. Manuscript, 2003.
4. Reinhard Diestel, Tommy R. Jensen, Konstantin Yu. Gorbunov, and Carsten Thomassen. Highly connected sets and the excluded grid theorem. *J. Combin. Theory Ser. B*, 75(1):61–73, 1999.
5. Rodney G. Downey and Michael R. Fellows. *Parameterized Complexity*. Springer-Verlag, New York, 1999.
6. David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000.
7. Markus Frick and Martin Grohe. Deciding first-order properties of locally tree-decomposable graphs. *J. ACM*, 48(6):1184–1206, 2001.
8. Neil Robertson and P. D. Seymour. Graph minors. VII. Disjoint paths on a surface. *J. Combin. Theory Ser. B*, 45(2):212–254, 1988.
9. Neil Robertson and P. D. Seymour. Graph minors. XII. Distance on a surface. *J. Combin. Theory Ser. B*, 64(2):240–272, 1995.
10. Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Combin. Theory Ser. B*, 62(2):323–348, 1994.