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EVALUATION OF DEFINITE INTEGRALS BY SYMBOLIC MANIPULATION

Paul S. Wang

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PROJECT MAC

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EVALUATION OF DEFINITE INTEGRALS BY SYMBOLIC MANIPULATION*

Abstract

A heuristic computer program for the evaluation of real definite integrals of elementary functions is described. This program, called WANDERER (WANg's DEfinite integRal EvaluatoR), evaluates many proper and improper integrals. The improper integrals may have a finite or infinite range of integration. Evaluation by contour integration and residue theory is among the methods used. A program called DELIMITER (DEfinitive LIMIT EvaluatoR) is used for the limit computations needed in evaluating some definite integrals. DELIMITER is a heuristic program written for computing limits of real or complex analytic functions. For real functions of a real variable, one-sided as well as two-sided limits can be computed. WANDERER and DELIMITER have been implemented in the MACSYMA system, a symbolic and algebraic manipulation system being developed at Project MAC, MIT. A typical problem in applied mathematics, namely asymptotic analysis of a definite integral, is solved using MACSYMA to demonstrate the usefulness of such a system and the facilities provided by WANDERER.

*This report reproduces a thesis of the same title submitted to the Department of Mathematics, Massachusetts Institute of Technology, in partial fulfillment of the requirements for the degree of Doctor of Philosophy, August 1971.

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CHAPTER I

INTRODUCTION

In recent years rapid advancement has taken place in the art of using modern electronic computers to facilitate symbolic mathematical computations. Computer based software systems for this purpose are generally known as symbolic manipulation systems. Such a system is designed to aid its users in handling mathematical expressions and functions. As the field of symbolic and algebraic manipulation grows stronger and more sophisticated so do the various algebraic manipulation systems. MACSYMA [16] is a recent redesign of earlier systems incorporating many new ideas and results developed in the field. The development of MACSYMA began at Project MAC, MIT, in early 1969. The effort has been guided by Professors Martin and Moses of MIT and involves a group of researchers and students including the author. Drawing on the past work of Martin [15], Moses [20] and Engelman [9], MACSYMA soon evolved into a system which extended the capabilities of automated algebraic manipulation systems in many new areas. Among these are the ability to compute limits of functions and the ability to evaluate definite integrals.

The limit concept is basic to mathematical analysis. Being able to compute limits automatically greatly increases the potential of a symbol manipulation system in doing analytical mathematics. In fact the evaluation of definite integrals is heavily dependent on the limit process, as is the expansion of functions in power series and many other mathematical problems. Although the computation of limits has been studied previously to some extent [11, 13], we describe a limit program called DELIMITER (DEfinitive LIMIT EvaluatoR) which is more powerful than previous programs. It is discussed in detail in chapter 2.

The problem of computing indefinite integrals symbolically by computers has been investigated rather thoroughly. First among the computer programs developed for this purpose was SAINT (Symbolic Automatic INTegrator) by Slagle in 1961. A more powerful program named SIN (Symbolic INtegrator) [28] was developed in 1967 by Moses. Theoretical work in this area include Richardson's undecidability result for a certain classes of integrals [22] and Risch's decision procedure for determining the existence of the indefinite integral of a member in the class of elementary functions [23, 24, 25, 26]. There is a comprehensive review by Moses on the progress during the past ten years in this particular area of symbol manipulation [21].

However, hardly any work has been done in the direction of definite integration. Reported here is a first definite integration program called WANDERER (WANg's DEfinite integRal EvaluatoR). Experiments with DELIMITER and WANDERER are the principal subjects of this thesis. WANDERER has been implemented in MACSYMA and makes use of many facilities provided in it. Some of these facilities are: input and two-dimensional output, simplification, solution of polynomial and systems of linear equations, canonical rational function simplification and the SIN [20] integration program.

Evaluation of definite integrals can sometimes be as easy as computing the indefinite integral of the given integrand then substituting in the limits of integration. WANDERER computes integrals of this kind by using those parts of SIN that have being implemented in MACSYMA. However, many interesting definite integrals are not obtainable in this manner. In some cases they may be improper integrals or their indefinite integrals do not exist. In other cases it is easier to evaluate the definite integral directly than to obtain the indefinite integral first.

A few limit and integration problems are listed below to give an indication as to the scope of WANDERER and DELIMITER. The results obtained by the programs can be found in chapters 2, 4 and 5.

$$\int_{-\infty}^{\infty} \frac{X^{2} + A\,X + B}{X^{4} + 10\,X^{2} + 9}\,dX \qquad \int_{0}^{\infty} \frac{X^{6}\,dX}{(X^{2} + X + 1)^{15/2}}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{SIN}(X)\,dX}{e^{2\,i\,X}\,(X^{2} + 1)} \qquad \int_{0}^{\infty} e^{-S\,T}\,X^{1/3}\,\mathrm{LOG}(X)\,dX$$

$$\int_{0}^{1} \frac{dX}{X^{2} - 3} \qquad \int_{3}^{4} \frac{dX}{X\,(X^{2} - 9)^{1/2}}$$

$$\int_{0}^{2\pi} \mathrm{COS}^{2}(X) - \mathrm{SIN}(X)\,dX \qquad \int_{1}^{\infty} \mathrm{LOG}^{1/2}(X)\,\,dX/\,X^{2}$$

$$\mathop{\mathrm{LIMIT}}_{X\to+\infty} X^{2}\,(4\,X^{4} + 5)^{1/2} - 2X^{4} \qquad \mathop{\mathrm{LIMIT}}_{X\to 0} (A\,X + 1)^{1/X}$$

$$\mathop{\mathrm{LIMIT}}_{X\to+\infty} e^{X\,(X^{2} + 1)^{1/2}} - e^{X^{2}}$$

Integrals with an infinite range are discussed in chapter 4, those with a finite range in chapter 5. Important routines and algorithms are collected in chapter 6.

An application of the MACSYMA system to an asymptotic analysis problem is illustrated in chapter 7. The purpose is to show how such a symbol manipulation system can be used to solve non-trivial problems that may occur frequently in applied mathematics. This application demonstrates the usefulness of the many facilities provided by MACSYMA and WANDERER.

A timing experiment has been conducted to check the performance of WANDERER and DELIMITER. The results are included in appendix C. Because of the limited character set of most computer consoles, some special symbols are needed to denote the frequently used mathematical functions, constants and operators. For example, * and ** are used to denote multiplication and exponentiation respectively. Appendix D contains a list of notations we shall use.

From Richardson's undecidability results [22], we have shown that the convergence of a class of integrals of elementary functions is recursively undecidable. The proof is in appendix E.

The remainder of this chapter serves as an introduction to algebraic manipulation systems. If the reader is familiar with such systems he may proceed directly to the next chapter.

An algebraic manipulation system is a collection of computer programs designed to facilitate the solution of mathematical problems. Such a system has the ability to handle both symbols and numbers. This capability to manipulate symbols mathematically is what differentiates these systems from the various computer subroutines which specialize in numerical analysis.

In using symbol manipulation systems such as these, one usually interacts with it in a time-sharing environment via a typewriter-like console. Normally, data and commands are typed in by the user. Results from the computer are sent back to the console for display. This arrangement attempts to provide a user the ease and flexibility of the pencil and paper he is so accustomed to, while permitting the computer to assist him in his algebra and formal deduction from one step to the next. If the manipulations involved are non-trivial, so much the better. Some of the manipulations such a system can provide are: GCD calculations, factoring, rational function arithmetic, matrix manipulation, solution of algebraic equations, solution of systems of linear equations, indefinite integration of elementary functions.

Let us take a closer look at MACSYMA as a representative model of other systems. MACSYMA receives inputs in the form of linear character strings typed by the user. FORTRAN-like notation is used for the input. For example,

The user types: 4*(X**2+X+1)/Y@

The @ sign signifies the end of a command string. As a result of this command, a two dimensional display is returned.

```
                          2
                      4 (X  + X + 1)
MACSYMA types:        --------------
                            Y
```

Inside MACSYMA, expressions are represented by list structures in a prefix notation (common to many systems). For example the expression above would be represented internally by something like

```
(TIMES 4 (PLUS 1 X (EXPT X 2)) (EXPT Y -1))
```

inside MACSYMA.

Other forms of internal representations are possible. The choice of internal representation is an important aspect of the design of a system. Of course more than one form of internal representation can be employed in a single algebraic manipulation system. In fact MACSYMA has a special internal representation for rational functions which is used to gain efficiency during certain polynomial and rational function manipulations.

One simple application of MACSYMA is factoring the polynomial

$$P(X) = X^{3} + 4\ X^{2}\ -11\ X - 30$$

A user who wants to factor P(X) using MACSYMA types:

```
FACTOR(X**3+4*X**2-11*X-33)@
```

This input command causes the factorization of P(X) over the integers. The output is

$$(X + 2)\ (X - 3)\ (X + 5)$$

While this problem may seem easy, factorization of polynomials of higher degree can be very difficult to do by hand. Indeed, algebraic manipulation systems can be most helpful when one wants to manipulate complicated functions and expressions. MACSYMA can carry out accurately with great speed: summation over indices, expansion of products and powers, calculation of large determinants, inversion of large matrices.

Applications in pure mathematics include computations in: number theory, group theory, Lie algebras and set theory.

Research in the field of symbol and algebraic manipulation has led to many new results. The fast GCD algorithm [4] and the finite field arithmetic polynomial factoring algorithm [2] are two examples. Continued work in this area will, hopefully, result in computer systems which are increasingly valuable to engineers and mathematicians.

CHAPTER II

DELIMITER

0. Introduction

The limit concept is fundamental to mathematical analysis. Basic concepts such as the rules of differentiation are derived from limiting processes. More complex problems such as improper integration, convergence of series, series expansion of functions and contour integrals, to name a few, also require the computation of limits in their solution process.

Therefore, one can expect that automatic computation of limits would greatly increase the capability of a symbolic mathematics system in doing analytical mathematics. The programs described below provide such a capability in the MACSYMA system [16].

Automating the computation of limits has been studied previously to some extent. Fenichel [11] discussed certain decidability problems of limits and provided, in the FAMOUS system, some basic routines for computing two-sided limits.

In his thesis, Iturriaga [13] worked on both one and two-sided limits. In addition to L'Hospital rule, he applied asymptotic analysis to quotients of polynomials essentially by replacing those polynomials by their leading terms. This can be done when the variable in the limit approaches infinity so that the leading term is asymptotically equivalent to the original polynomial. Moreover, he discussed limits of sequences.

DELIMITER is more powerful than either Fenichel's or Iturriaga's limit program. L'Hospital's rule is a basic method used for indeterminate forms. In addition, this program employs a fast routine for limits of rational functions. It has an efficient algorithm for a class of expressions called RP-expressions which involve radicals of polynomials. It also applies the method of reducing complicated expressions by replacing subexpressions with asymptotically equivalent expressions. The method of comparing orders of infinity of expressions and several other heuristic methods are used. In some cases, power series expansions are employed to obtain the limits.

This chapter is based on a paper by the author [29] presented at The Second Symposium on Symbolic and Algebraic Manipulation, Los Angeles, March, 1971.

1. Definitions and Symbols

DELIMITER is designed for finding the limit of a single-valued function f(X) of a real or complex variable X as X approaches some limit point. One-sided limits can be computed if X and f(X) are real-valued. The classes of functions allowed by this program include rational, radical, logarithmic and exponential functions, and also the trigonometric functions SIN, COS and TAN, and the hyperbolic functions SINH, COSH and TANH. The general form of a user command is

```
LIMIT(expression,variable,value,direction)@
```

with the fourth argument optional. PLUS as a fourth argument indicates the one-sided limit from above, MINUS from below. The absence of the fourth argument indicates no restriction on the direction of approach. One-sided limits are not allowed for complex-valued functions or limit points.

If f(X) is not continuous at X=a, the two-sided limit, LIMIT(f(X),X,a), does not exist. In such a case the symbol UND is returned by the program as an answer.

The following are some symbols used by this program.

ZERO+	0+	ZERO-	0-
UND	undefined		
IND	indefinite but bounded		
INFINITY	the point at infinity in the complex plane		

INF and MINF will continue to indicate respectively positive and negative infinity. Here are some examples of how these symbols are used internally.

LIMIT(COS(X),X,%PI) = -1+ ZERO+

LIMIT(SIN(X),X,INF) = IND

LIMIT(1/SIN(X),X,INF) = UND

LIMIT(1/(X-%I),X,%I) = INFINITY

The direction from which a limit point is approached is important. By use of ZERO+ and ZERO-, simplification rules such as

1/(-1 + ZERO+ + 1) ⟶ INF

and

1/(-1 + ZERO- + 1) ⟶ MINF

are possible in DELIMITER.

2. Basic Rules and Outline of Algorithm

Some of the rules used in this program are the so-called "trivial" ones for limits of continuous functions, namely, the limit of a sum is the sum of the limits, etc. There is a limit routine for each of the functions SIN, COS, TAN, SINH, COSH, TANH and LOG (base %E). For other functions the rule

LIMIT(f(g(X)),X,L) = LIMIT(f(Y),Y,LIMIT(g(X),X,L))

is used.

For one-sided limits where the variable approaches a point other than 0, a change of variable is made to bring the point to the origin. For example, the limit

LIMIT(f(X),X,a,PLUS)

is converted to

LIMIT(f(a+Y),Y,ZERO+)

There is, of course, a complete set of rules governing simplification of the new symbols. They are all of the following nature

```
IND/INF = 0              -1/INF = ZERO-
- ZERO+ = ZERO-          - INF = MINF
1/INFINITY = 0           -1/INFINITY = 0
MINF**2 = INF            (ZERO+)+ZERO- = 0
LOG(C+ZERO+) = LOG(C) + ZERO+
```

We will now briefly outline the algorithm, and then go into the details of some of the component routines. Let us consider

LIMIT (E(X),X,L)

where L is any number or symbol including INFINITY, INF, ZERO+ and ZERO-. If L = MINF, it is set to INF by a change of variable Y=-X. Upon receiving the arguments, namely E(X), X and L, DELIMITER checks whether L=INFINITY or E(X) involves %I. If either or both is true then a global indicator CPLX is set to the value TRUE which indicates that the given limit is to be evaluated over the field of complex numbers. Otherwise CPLX is set to the value FALSE which signifies a limit problem over the reals. If CPLX=TRUE, all use of INF and MINF are replaced by INFINITY and the notion of approaching a limit point from one side is no longer valid.

The following diagram is a simplified overview of DELIMITER.

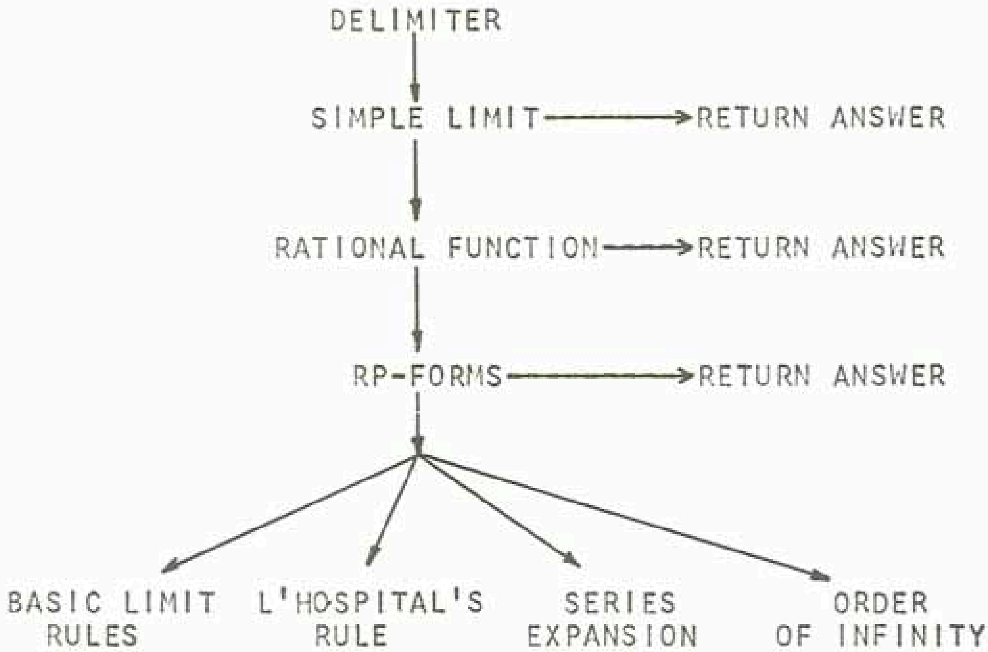


Fig. 1

As indicated in fig. 1, DELIMITER has four stages. Results of simple limit problems are immediately returned by the first stage. Simple limits include two cases: 1) E is independent of X. 2) E(X) = X. In the second stage, a fast algorithm is used to obtain limits of rational functions. The third stage employs an algorithm for limits of RP-forms. If the first three stages do not produce an answer, the given limit problem enters the fourth stage which contains a variety of methods including the four principal ones shown in the figure. In this stage the program will first try to extract the numerator N(X) and the denominator D(X) of E(X) so that E(X) = N(X)/D(X). If both N(X) and D(X) are RP-forms the limit is computed by an algorithm specially for these forms (see section 5 for definition and algorithm). Otherwise the rule

lim E(X) = lim N(X) /lim D(X)

is applied. If both lim N(X) and lim D(X) are 0 or infinite we have an indeterminate form. In this case L'Hospital's rule will be applied to E(X) with one exception: when L is INF and lim N(X) and lim D(X) are both infinite and both N(X) and D(X) contain exponential functions of X which tend to INF as X approaches L. In this case the method of comparing orders of infinity, to be described shortly, is applied. If DELIMITER can not find an answer indication of failure will be returned.

Sometimes the program needs to know the value range of a symbolic parameter in order to compute the limit. In such cases, the program will query the user at his console.

It is important to note that these programs do not store any table of limits Thus every limit obtained is a result of computation.

Other methods available include : change of variable, discontinuity tests and analysis of the behavior of a function near a finite point. A flowchart in appendix A presents the flow of control in a more complete manner.

3. The Limit of a Rational Function

Let P(X) and Q(X) be polynomials in X and possibly other variables and

E(X) =P(X)/Q(X),

n =degree(P)-degree(Q),

c =leading coefficient(P)/leading coefficient(Q),

Consider LIMIT(E(X),X,L) where E(X) or X may be complex valued. The following algorithm is used to compute the limit.

(1) If L is finite then,
If P(X) = Q(X) = 0 then set P and Q according to
P(X)= P(X) /(X-L), Q(X)= Q(X) /(X-L),
by long division, then go to step (1);
if Q(L) is not 0 then the answer is E(L);
otherwise, the answer is INFINITY if CPLX=TRUE and, if CPLX is FALSE, the result is INF or MINF depending on the direction from which Q(X) approaches 0 as X approaches L, and of course the sign of P(L).

(2) If L is infinite,

if n=0, the answer is c;

if n < 0, then if CLPX = TRUE, answer is 0, but if CPLX is FALSE then the answer is ZERO+ provided the sign of c*L**n is + and ZERO- in case the sign of c*L**n is -;

otherwise, (n > 0) the answer is INFINITY if CPLX=TRUE, and the answer is INF or MINF, depending on the sign of c*L**n, if CPLX=FALSE.

4. The Limit of a Quotient of RP-forms

An RP-form is defined as an expression obtainable by combining polynomials and positive rational number powers of polynomials using the operators +, -, *. Here are some examples of RP-forms,

$$f(X) = SQRT(4\ X^{3} + 5\ X) + (X + 1)^{1/3} + X + C$$

$$g(X) = A\ X^{2/3}\ SQRT(X+1) + X$$

Note that the definition of RP-forms does not allow nested radicals of X. For instance the following is not an RP-form.

$$(X + SQRT(X))^{2/3}$$

Now consider the limit problem,

LIMIT(E(X),X,L), E(X) =N(X)/D(X)

where N(X) and D(X)≠0 are RP-forms and at least one of them is irrational. Note that N(X) or D(X) can be a constant.

Let us define the operators EXPO and COEF by

EXPO(N(X))

=highest exponent of X in N(X),

COEF(E,N(X))

=the coefficient of E in N(X).

For example,

EXPO(g(X)) =7/6,

COEF(X**(3/2),f(X)) =2

EXPO is obtained by a simple special purpose routine; COEF is computed by using the RATCOEF routine in the rational function package of MACSYMA [16].

Now the limit is computed by the following algorithm.

(1) If L =MINF, X is replaced by -X, L by INF.

(2) If L =INF, the following asymptotic analysis is made.

i) compute a =EXPO(N(X)), b=EXPO(D(X)).

ii) Let N1(X) and D1(X) be N(X) and D(X) with polynomials under radicals replaced by their leading terms, respectively. Compute

a'=EXPO(N1), b'=EXPO(D1).

iii) If either $a \neq a'$ or $b \neq b'$, then it is usually sufficient to replace each radical in E(X) by the first few terms in the series expansion of the radical about X=INF, obtaining a rational function R(X). The answer is

LIMIT(R(X),X,INF).

Otherwise, compute c=COEF(X**a,N1)/COEF(X**b,N2). If b>a, return the answer c*ZERO+ if a>b, return the answer c*INF, otherwise, return the answer c.

(3) If L = INFINITY then

Carry out steps (2-i) and (2-ii). If a≠a', b≠b' or a=b, the answer is UND. Otherwise, if a > b the answer is INFINITY, while if a < b the answer is 0.

(4) If L is finite, compute N(L) and D(L), then

If N(L)=D(L)=0, apply L'Hospital's rule. Since L'Hospital's rule can not succeed in case each term in N(X) and D(X) has a branch point at X=L, that is, N(X) and D(X) have a common factor (X-L)**(a/b) with a and b positive integers. In this case the factor is removed from N(X) and

D(X) before using L'Hospital's Rule.

If N(L)=0 and D(L)≠0,
the answer is 0 if CPLX=TRUE, otherwise the behavior of D(L)*N(X) at L is examined to decide whether 0, ZERO+ or ZERO- is the answer.

If D(L)=0 and N(L)≠0,
the answer is INFINITY if CPLX =TRUE, otherwise the behavior of N(L)*D(X) at L is examined to decide whether INF or MINF is the answer.
Otherwise, the answer is E(L).

For large X the series expansion

$$\mathrm{SQRT}(X^2 + 1) = \mathrm{SQRT}\left(\frac{1}{X^2} + 1\right) X$$

$$= X \left(1 + \frac{1}{2\,X^2} - \frac{1}{8\,X^4} + \ldots\right)$$

$$= X + \frac{1}{2\,X} - \frac{1}{8\,X^3} + \ldots$$

is convergent. It is clear that this expansion method can be applied to an arbitrary polynomial to a positive fractional power. This fact establishes the validity of step (2-iii).

5. Indeterminate Forms and L'Hospital's Rule

L'Hospital's rule is applied to an indeterminate form E(X) of the form 0/0 or INFINITY/INFINITY. It first calculates N'(X) and D'(X), then,

If D'(X)=0, L'Hospital's rule fails, otherwise, simplify N'(X)/D'(X) and evaluate

LIMIT (N'(X)/D'(X),X,L).

The expression N'(X)/D'(X) can be more complex than E(X) and successive application of this rule may lead nowhere. Therefore the number of times this rule is called successively is counted and the relative complexity of N'(X)/D'(X) to E(X) is tested to decide whether to continue this approach. Our criterion of complexity is based on the number of distinct nonrational components of an expression. If this number grows for three consecutive times, the application of L'Hospital's rule is halted.

The indeterminate form 0*INFINITY is transformed to either 0/(1/INFINITY) or INFINITY/(1/0), depending on which is simpler, before applying L'Hospital's rule. Other indeterminate forms such as 1**INFINITY, INF**0 and 0**0 are handled by the logarithmic reduction :

```
F(X)**G(X) = %E**(G(X)*LOG(F(X))).
```

Theoretically the indeterminate form (INF - INF) can always be reduced to 0/0 by rewriting it as

```
(I)              (1/INF - 1/INF)/(1/INF*INF),
```

but this method often makes the expression much more complicated. It is useful, though, for expressions involving trigonometric functions as can be seen in

```
          1     1
    LIMIT(- - ------,X,0,PLUS) = 0.
          X   SIN(X)
```

DELIMITER transforms the given expression in this example to

```
SIN(X)-X
--------
X SIN(X),
```

then applies L'Hospital's Rule to obtain the answer 0. Therefore, the method (I) is used for expressions involving trigonometric functions. Other types of expressions can be dealt with more readily by comparing degrees of infinity of the subexpressions or by series expansion.

6. On the Sign of Infinity

If a function f(X) is discontinuous at X=a, another function g(f(X)) is not necessarily discontinuous at a. For example

$$\mathrm{LIMIT}((A\ X + 1)^{1/X}, X, 0) = \%E^{A}$$

Therefore, discontinuity may be encountered in the course of computing a limit of a continuous function. Consider 1/E(X) with E(a)=0 and the limit problem

LIMIT(1/E(X),X,a),

If CPLX = TRUE, the answer is INFINITY. Otherwise, a routine named BEHAVIOR is used to analyze the behavior of E(X) near X=a. If E(X) approaches 0 from above or below as X approaches a, then the answer is INF or MINF, respectively. Otherwise, the answer is INFINITY.

The BEHAVIOR routine uses differentiation to analyze the behavior of a function near a point by investigating its slope or concavity at the point. This routine is also used by other programs. One of these is the program which computes limits of the trigonometric function tangent. In case the argument of TAN approaches %PI/2, say, it is important to know the direction of approach.

7. Heuristics for Comparing Orders of Infinity

Let f(X) and g(X) be two real-valued functions which become positively infinite as X approaches INF. We define the symbol >> by

Definition : g(X) >> f(X) if

LIMIT(f(X)/g(X),X,INF) =0.

Using this symbol we can make the following brief table.

... %E**%E**X>>%E*X>>X**n>>log(X)>>log(log(X)) ...

Many limit problems with the variable approaching INF can be solved very efficiently by using this concept. Some of these problems can be difficult to solve by other methods. For example, L'Hospital rule fails to compute the limit

LIMIT((%E+1)**X**2/%E**X,X,INF),

while the answer is obviously INF. We incorporate this concept by a routine, STRENGTH. It can classify the order of infinity of the argument according to the following rules, STRENGTH(c)=(c)

STRENGTH(LOG(R(X)))= LOG

STRENGTH(A**B(X))= EXP

STRENGTH(P(X)**(m/n))= m*d/n

where c is independent of X, m and n are positive integers, d the degree of the polynomial P(X). Note the STRENGTH of a constant is denoted by that constant inside parenthesis and the STRENGTH of a polynomial is a constant. B(X) and R(X) can be any functions in X, except those which can cause the relevant argument of STRENGTH to be simplified into one of the other three cases. We can assume A to be %E for if A is any other expression it is always possible to change the base to %E. STRENGTH of a sum is the maximum of the strengths of the terms in the sum.

There is a basic comparison routine which knows the relations between the strengths of functions,

EXP>>a>>LOG>>(b) and c>>d if c>d

where a, b, c, and d are constants.

Let N(X) and D(X) be two products satisfying

LIMIT(N(X),X,INF)=INF

and

LIMIT(D(X),X,INF)=INF

and no factor in N(X) or D(X) is a sum or a trigonometric function. To compare the relative order of infinity of N(X) and D(X), the following method is used. The answer 1 indicates N(X)>>D(X), -1 indicates D(X)>>N(X), 0 no decision.

Algorithm COMPARE(N(X),D(X)):

1) Remove from N(X) and D(X) any common factors.

2) Apply the STRENGTH routine to each factor of N(X) and obtain the maximum strength SN.

3) Do the same to D(X) and obtain its maximum strength SD.

4) If SN>>SD the answer is 1, if SD<<SN the answer is -1.

Otherwise,

i) Let
N1(X)=product of all factors with strength SN in N(X).
D1(X)=product of all factors with strength SD in

D(X).

ii) If SN=SD=some constant, set

SN=STRENGTH(N1) and SD=STRENGTH(D1).

If SN>>SD, the answer is 1, if SN<<SD the answer is -1. If both N1(X) and D1(X) are polynomials in X, the answer is the leading coefficient of the polynomial N1(X) - D1(X). Otherwise if the variable INDICATOR has the value 1, this means each of N(X) and D(X) is an exponent of an exponent of an exponential expression and they come as a result of recursive calls to COMPARE. For example N(X) may come from %E**%E**N(X). In this case, the algorithm goes as follows. First INDICATOR is set to 0 then E=LIMIT(N(X)-D(X),X,INF) is computed and the answer is 1 if E>0, -1 if E<0, 0 otherwise.

If SN=SD=EXP,

The exponents A(X) and B(X) of N1(X) and D1(X) are computed. That is N1(X) =%E**A(X) and D1(X) =%E**B(X). If both A(X) and B(X) are polynomials in X, the answer is the leading coefficient of the polynomial A(X) - B(X) if this polynomial is not a constant, 0 if it is. If A or B is a sum, A is set to LIMIT(A(X)-B(X),X,INF) and the answer is 1 if A=INF, -1 if A=MINF, 0 otherwise. In case neither A nor B is a sum, if both A and B are of strength EXP, set the variable INDICATOR to 1. The answer is obtained by evaluating COMPARE(A1(X),B1(X)), where A(X) = %E**A1(x) and B(x) = %E**B1(X).

If SN=SD=LOG,

If N1(X) is LOG(f(X)) and D1(X) is LOG(g(X)) then set SN to STRENGTH(f(X)), SD to STRENGTH(g(X)). Now the answer is 0 if both SN and SD are constants, 1 if SN>>SD, -1 if SN<<SD. The answer is 0 otherwise.

We shall next discuss the indeterminate form (INF-INF) and see how COMPARE can be used in such situations. Let

$$F(X) = \sum_{i=1}^{n} f_i(X)$$

where

$$\mathrm{LIMIT}(f_i(X),X,\mathrm{INF}) = \mathrm{INF} \text{ or } \mathrm{MINF}$$

for i=1,2,...n.

LIMIT(F(X),X,INF) becomes indeterminate when there exists i and j such that

$$\mathrm{LIMIT}(f_i(X),X,\mathrm{INF}) = \mathrm{INF}$$

and

$$\mathrm{LIMIT}(f_j(X),X,\mathrm{INF}) = \mathrm{MINF}$$

The method of orders of infinity can sometimes be applied to this problem. The algorithm used is as follows.

The program forms two lists L1 and L2 of the f_i's which go to INF and MINF respectively. Then from each of L1 and L2 a member of maximum order of infinity can be obtained. Let these be r(X) in L1 and s(X) in L2. Now if r(X) >>s(X) the answer is INF, if s(X)>>r(X) the answer is MINF. Otherwise, the problem can be very difficult. However the program tries to evaluate

INF*LIMIT(F(X)/s(X),X,INF),

which sometimes produces an answer.

Now we will follow the major steps of the solution of a problem in a more detailed manner. Let

$$A(X) = \%E^{X*SQRT(X^{2} + 1)}$$

$$B(X) = \%E^{X^{2}}$$

Consider the problem LIMIT(A(X)-B(X),X,INF). First the indeterminate form (INF-INF) is encountered through the following steps.

LIMIT(A(X)-B(X),X,INF)

```
LIMIT(A(X),X,INF)-LIMIT(B(X),X,INF)

LIMIT(A(X),X,INF) =%E**INF =INF

LIMIT(B(X),X,INF) =%E**INF =INF

INF-INF
```

Then COMPARE is applied to A(X) and B(X), only to arrive at no conclusion about the relative order of infinity of A(X) and B(X).

```
COMPARE(A(X),B(X))

STRENGTH(A(X))=EXP

STRENGTH(B(X))=EXP
                   2       2
COMPARE(X*SQRT(X  + 1),X )
                    2
STRENGTH(X*SQRT(X  + 1)) = 2
              2
STRENGTH(X ) = 2
```

Now the problem is converted to the following form and the answer is INF.

```
INF*(LIMIT(A(X)/B(X),X,INF)-1)

                 2          2
LIMIT(X*SQRT(X  + 1) - X ,X,INF) = 1/2
```

```
INF*(%E**(1/2)-1)

INF
```

8. Examples

A number of limit problems solved by DELIMITER are included in this section. They are presented in the form of actual inputs and outputs of the MACSYMA system. Lines labelled (CI) are input or command lines and (DI) output lines or answers.

```
(C1) LIMIT(X**LOG(1/X), X, INF)@
(D1)                              0
(C2) (COS(X)-1)/( %E**X**2 -1)@
                              COS(X) - 1
(D2)                          ----------
                                  2
                                 X
                              %E    - 1
(C3) LIMIT (D2,X,0)@
                                   1
(D3)                             - -
                                   2
(C4) (1+A*X)**(1/X)@
                                      1/X
(D4)                         (A X + 1)
(C5) LIMIT(D4,X,0)@
                                  A
(D5)                            %E
(C6) X**2*(4*X**4+5)**(1/2)-2*X**4@
```

```
(C6) X**2*(4*X**4+5)**(1/2)-2*X**4@

                         2          4         4
(D6)                    X  SQRT(4 X  + 5) - 2 X

(C7) LIMIT(D6,X,INF)@
                                       5
(D7)                                   -
                                       4
(C8) 1/X-1/SIN(X)@
                                     1     1
(D8)                                 - - ------
                                     X   SIN(X)
(C9) LIMIT(D8,X,0,PLUS)@

(D9)                                   0

(C10) %E**(X*(X**2+1)**(1/2))-%E**X**2@

                                 2              2
                           X SQRT(X  + 1)      X
(D10)                   %E                - %E

(C11) LIMIT(D1,X,INF)@

(D11)                                   INF

(C12) (%E**X+X*LOG(X))/(%E**(X**3+1)**(1/2)+LOG(X**4+X+1))@


                                X
                              %E  + X LOG(X)
(D12)               --------------------------------
                                                3
                           4              SQRT(X  + 1)
                    LOG(X  + X + 1) + %E

(C13) LIMIT(D4,X,INF)@

(D13)                          0

(C14) 1/(X**3-6*X**2+11*X-6)@
```

```
                            1
(D14)           -------------------
                 3      2
                X  - 6 X  + 11 X - 6

(C15) LIMIT(D14,X,2,MINUS)@

(D15)                      INF

(C16) (X*SQRT(X+5)+1)/(SQRT(4*X**3+1)+X)@

                 X SQRT(X + 5) + 1
(D16)            -----------------
                          3
                 SQRT(4 X  + 1) + X

(C17) LIMIT(D16,X,INF)@
                             1
(D17)                        -
                             2

(C18) TAN(X)/LOG(COS(X))@

                                TAN(X)
(D18)                        -----------
                             LOG(COS(X))

(C19) LIMIT(D18,X,%PI/2,MINUS)@

(D19)                         MINF
```

```
(C20) (Z-%I*%PI/2)*Z*(Z-2*%PI*%I)/(SINH(Z)-%I)@

                                                %PI %I
                    Z (Z - 2 %PI %I) (Z - ------)
                                                   2
(D20)               -----------------------------
                               SINH(Z) - %I

(C21) LIM(DIFF(D20,Z),Z,%I*%PI/2)@

(D21)                          - 2 %PI
```

CHAPTER III

WANDERER-AN INTRODUCTION AND OUTLINE

1. Introduction

The present chapter and the three subsequent ones are devoted to a complete description of WANDERER, a heuristic program for evaluating definite integrals over a real range.

WANDERER cannot solve all definite integration problems, but it can certainly obtain solutions to a large number of integrals. Many examples presented are actual problems taken out of graduate text books [1, 5]. The basic approach, in WANDERER, for evaluating an integral is by complex contour integration and residue techniques. This is a very general method particularly useful in evaluation of infinite integrals. Some other methods available to WANDERER are: substitution, integration by parts, differentiation with respect to a parameter, pattern recognition and table look-up, finite-to-infinite conversion, introduction of a parameter and partition and transformation of the range of integration.

The user's command in MACSYMA for definite integration is the key word DEFINT. DEFINT takes four arguments as in

DEFINT(exp, var, a, b)

where exp is the integrand, var the variable of integration, a and b the lower and upper limits of integration. There is no restriction as to what types of integrands are allowed as input, as long as they remain elementary functions. However, WANDERER requires the integrand to be finite in the integration range except possibly at the end points a and b.

A few special symbols are used in WANDERER for some branches of the logarithm function. They are listed in appendix D.

2. Outline of Approach

Before going into the details of methods and algorithms for evaluation of the many types of definite integrals which will be discussed in the next three chapters, a quick look at the whole picture with emphasis on the flow of control is in order.

WANDERER is a heuristic program which computes definite integrals by trying to apply one or more of the methods or algorithms built into it. The clues as to which of the methods to apply are obtained by examining the range of integration and the form of the integrand.

Fig. 1 on the next page serves as a simple outline of the flow of control in WANDERER.

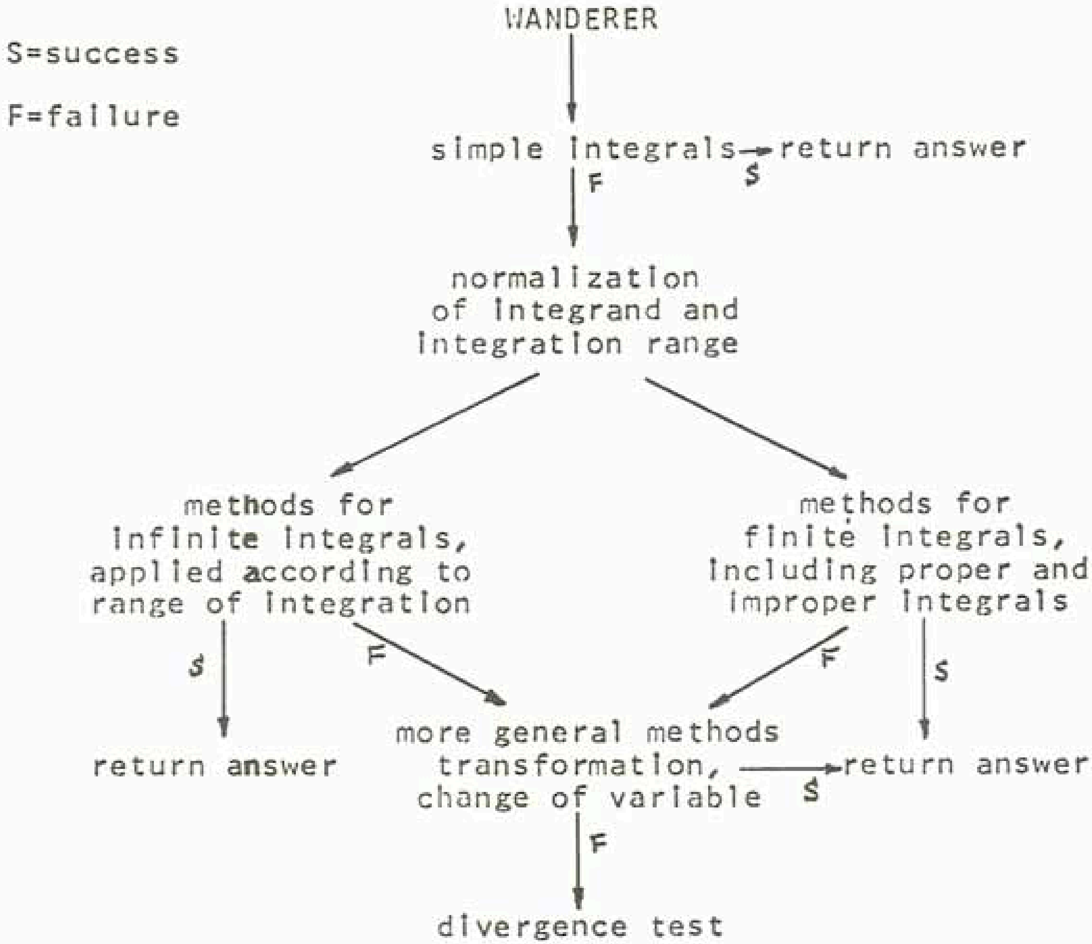


Fig. 1

As indicated in fig. 1. WANDERER can be said to have five stages. In the first stage simple integrals are computed which include cases such as a constant integrand, an integral with equal upper and lower limits of integration and, if the variable of integration is X, integral with integrands which are polynomials in X or %E**X. The normalization stage will make sure, by transformation if necessary, that (1) the lower limit is less than the upper limit, and (2) constant factors, if any, in the integrand are removed to be multiplied into the final answer of the integral.

The third stage comprises most of the methods that will be described in chapters 4 and 5 and is the work-horse of the whole program. Many methods in this stage are grouped according to the integration range in which they are appropriate. Finite integrals of rational functions are transformed into infinite integrals which will be evaluated. For other finite integrals, an effort is first made to determine whether they are improper integrals. The absolute divergence of a finite improper integral is tested before any attempt at evaluation. If the third stage fails to obtain any results, the given problem enters the fourth stage which contains more general methods appropriate to various types of integrands. A transformation or change of variable is often done in this stage. When WANDERER runs out of methods, the convergence of the given integral will be challenged. In case it is divergent, WANDERER will so indicate in the output. Otherwise, indication of failure will be returned. It may be argued that tests for convergence should be conducted before evaluation, but this is not necessary because WANDERER is so designed that whenever it produces an answer, the given integral is convergent. That is to say each individual method has its own convergence and divergence conditions built in.

The limit program described in the previous chapter is used whenever a limit computation is needed in WANDERER. It may often be required in changing the variable of integration, computing residues and testing for convergence or divergence.

3. Outline of the Computation of Residues

One of the most powerful methods in evaluation of definite integrals is the use of contour integration through residue calculations. The importance of contour integration and the residue theory in the sequel warrants a brief summary of relevant facts from complex analysis [5].

(i) The Cauchy Integral Theorem: Let D be a simply connected domain and let f(Z) be analytic in D. Let C be a closed contour in D. Then

$$\int_C f(Z)\, dZ = 0$$

A simply connected domain is intuitively an open set of points with no holes or cuts in its interior. For example the disk $|Z| < 0$ is simply connected while the annulus $1 < |Z| < 2$ is not. The Cauchy integral theorem is one of the most important tools in complex analysis. It is by use of this theorem that path of integration can be deformed. More precisely, if C1 and C2 are two different curves in D leading from the same starting point v to the same end point w, as shown in fig. 2,

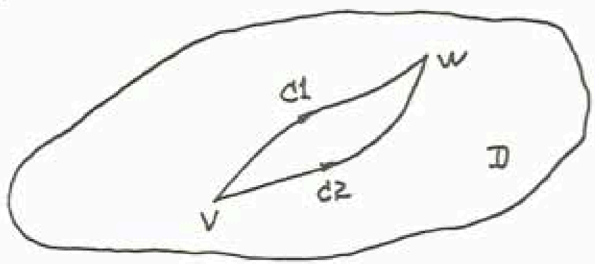


Fig. 2

and f(Z) is analytic in D, then

$$\int_{C1} f(Z)\, dZ = \int_{C2} f(Z)\, dZ.$$

Therefore, the contour C1 can be deformed into C2. The above is an immediate consequence of Cauchy's integral theorem which gives

$$\int_{C1} f(Z)\, dZ - \int_{C2} f(Z)\, dZ = 0$$

(ii) Definition of Residue: Let f(Z) be analytic in the punctured neighborhood of a point p. Let Cp be a small positively oriented circle with center at p. Then the residue of f at p is defined as

$$\frac{1}{2\pi i} \int_{C_p} f(Z)\, dZ$$

The value of this integral is the coefficient of the term

$$(Z - p)^{-1}$$

in the Laurent expansion of f about the point p, so this coefficient can also be regarded as the residue of f at p.

(iii) The Residue Theorem: Let C be a simple closed positively oriented contour. Let D be a simply connected domain containing C and its interior. Let A1, A2, . . ., An be points inside C. Except for isolated singularities at A1, A2, . . ., An, let f(Z) be an analytic function in D. Then

$$\int_C f(Z)\, dZ = 2\pi i \sum_{j=1}^{n} (\text{Residue of } f \text{ at } Aj)$$

The algorithm for residue computation used in WANDERER will be described briefly here. Detailed discussion of this algorithm can be found in sect. 6-1.

Let f(Z) be a function of a complex variable Z, analytic everywhere in a domain D except for a number of poles in D. Suppose F(Z) can be written in the form

$$f(Z) = U(Z)/V(Z)$$

such that U(Z) is analytic in D. This means, that poles of f(Z) are zeroes of V(Z). Suppose p is a pole of order m of f(Z), the residue of f(Z) at p is computed by the following algorithm.

RESIDUE ALGORITHM:

If m = 1, compute as answer

$$U(p)/V'(p)$$

otherwise, if V is a polynomial,

(i) Set V to the quotient of V(Z)/(Z - p)**m which is computed by long division.

(ii) Return the result computed from

$$\frac{1}{(m-1)!}\left(\frac{d}{dZ}\right)^{m-1}\frac{U(Z)}{V(Z)}\Bigg|_{Z=p}$$

otherwise, ($m > 1$, V not a polynomial) compute and return as answer the limit

$$\lim_{Z \to p} \left[\frac{1}{(m - 1)!} \left(\frac{d}{dZ}\right)^{m-1} (Z - p)^{m} f(Z) \right]$$

DELIMITER is used in obtaining such a limit. Methods for infinite integrals are discussed in the next chapter. Those for finite integrals are included in chapter 5. Important algorithms and algorithms common to many methods are detailed in chapter 6.

CHAPTER IV
SYMBOLIC COMPUTATION OF INFINITE INTEGRALS

0. Introduction

The aim of this chapter is to give a detailed account of the methods employed by WANDERER for improper integrals with an infinite range. Contour integration and residue computation play a very important role in many of the methods. The algorithm for integrals with rational integrands, which will be discussed first, is the most complete. Sections are formed according to the function types of the integrand and ordered roughly in increasing complexity. Quite a few examples are included, some of them with references to books or integral tables indicated.

1. Infinite Integral of a Rational Function

In this section methods for evaluating infinite integrals of a rational function will be presented. The range of integration is from either 0 or minus infinity to infinity. Other possible infinite ranges such as (a INF), (MINF a), where a is finite, can be converted to (0 INF) by a change of variable in the integral.

First let us consider

$$L = \int_{-\infty}^{\infty} R(X)\, dX, \qquad R(X) = P(X)/Q(X)$$

where P(X) and Q(X) are polynomials in X over the field of complex numbers.

WANDERER requires $\deg(Q(X)) - \deg(P(X)) \geq 2$ to insure the convergence of the given integral. If R(Z) has no real poles, then the integral L can be computed by evaluating the contour integral

$$J = \int_{C} R(Z)\, dZ$$

around a familiar semi-circular contour in the upper complex Z-plane (fig. 1). One can easily prove that L = J as r tends to infinity.

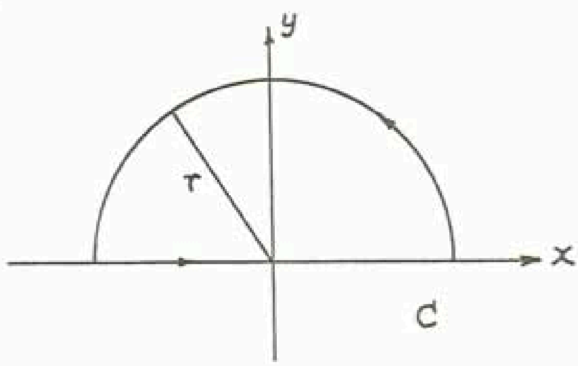


fig. 1

Therefore

(1) L = 2 %PI %I($\sum$ Res(R) over poles of R inside C)

The residue subroutine outlined in the previous chapter is used to compute the residues needed in this formula. If R(Z) has poles on the real axis then L is divergent. However, if the real poles of R are all simple then the Cauchy Principal Value of the integral L exists, and can be obtained by indenting the contour C at these singular points on the real axis. For such integrands WANDERER will compute this principal value, which is given by

(2) (P)L = 2 %PI %I($\sum$ Res(R) over poles of R inside C)

+ %PI %I(Res(R) over simple real poles of R),

as an answer to the integral L. (P)L stands for the principal value of L. Whenever the answer is a principal value, the message PRINCIPAL will be sent to the user first.

Here is an example solved by WANDERER

```
(C1) (X**2+A*X+B)/(X**4+10*X**2+9)@
                       2
                      X  + A X + B
(D1)                 --------------
                      4       2
                     X  + 10 X  + 9

(C2) DEFINT(D1,X,MINF,INF)@
                      %PI B + 3 %PI
(D2)                  -------------
                           12
```

The expression D1 has four simple poles: X = %I, -%I, 3 %I and -3 %I. Using formula (1), only residues at X=%I and X=3 %I need be computed. The residues are

at X=%I	(B-1+A %I)/(16'%I)
at X=3 %I	-(B-9+3 A %I)/(48 %I)

These values are computed by the residue algorithm given in the previous chapter.

The full algorithm depends on finding the poles of R. How this is done will be discussed in detail after considering the next integral.

$$K = \int_0^\infty R(X)\ dX$$

If R(Z) has no real pole which is positive or zero, this integral can be evaluated by integrating the following contour integral [30]

$$J = \int_C f(Z)\ dZ, \qquad f(Z) = \mathrm{PLOG}(-Z)\ R(Z)$$

where PLOG denotes the principal branch of the LOG function, i.e., the imaginary part of LOG(-Z) lies between %PI and - %PI. The contour C consists of circular arcs of radii a, b, and two straight lines joining their end-points as shown in fig. 2.

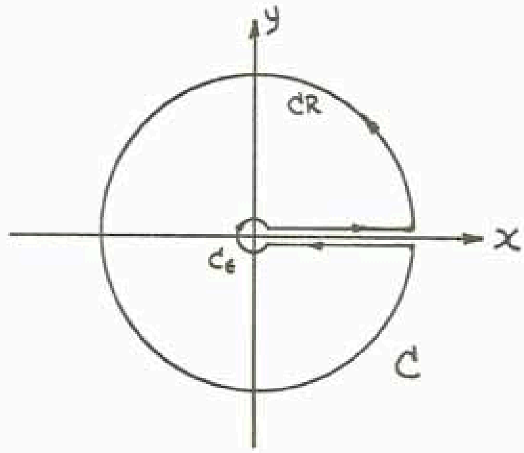


fig. 2

One can verify that contributions from the two circular arcs of C vanishes in the limit. Thus,

$$J = \int_0^\infty \mathrm{PLOG}(-X)\ R(X)\ dX + \int_\infty^0 \mathrm{PLOG}(-X\ \%E^{2\ \%PI\ \%I})\ R(X)\ dX$$

$$= \int_0^\infty \mathrm{PLOG}(X)\ R(X)\ dX - \int_0^\infty \mathrm{PLOG}(X)\ R(X)\ dX - 2\ \%PI\ \%I \int_0^\infty R(X)\ dX$$

$$= -\ 2\ \%PI\ \%I\ K$$

Hence the value of the integral K is given by

(3) $K = -(\sum \mathrm{Res}(f)$ over poles of R inside C)

Therefore, the value of K is -1 times the sum of residues of LOG(-Z)R(Z) in the complex Z-plane cut along the positive real axis.

It may be of some interest here to mention that a theorem in |30| gives a value for the integral K which differs from the correct formula (3) by a sign.

Now suppose R(Z) has poles which are real and positive, then the integral K is divergent. But its Cauchy principal value exists if every such pole is of order 1. This value is given by -1 times the sum of residues of LOG(-Z)R(Z) in the entire Z-plane punctured at Z=0, as given in

(4) $(P)K = -(\sum \mathrm{Res}(f)$ over poles of R).

K diverges if R(Z) has a pole at Z=0. The following examples were computed by WANDERER. For ease of reference, expression (D1) is repeated here

(D1)
$$\frac{X^2 + A\ X + B}{X^4 + 10\ X^2 + 9}$$

(C3) DEFINT(D1,X,0,INF)@

(D3)
$$\frac{2\ \%PI\ B + 6\ LOG(3)\ A + 6\ \%PI}{48}$$

The answer in (D3) is obtained by locating the four simple poles of (D1) and applying formula (3).

(C4)1/(X**2+X+1)@

(D4)
$$\frac{1}{(X^2 + X + 1)}$$

(C5)DEFINT(D6,X,0,INF)@

$$\frac{2\ \%PI}{3\ SQRT(3)}$$

Now let us turn to the problem of finding the poles of R(Z). The SOLVE [16] program in MACSYMA knows how to solve a number of types of equations. This routine is used in obtaining the locations and multiplicities of the zeros of Q(x), the denominator of R(X). The SOLVE program does this by factoring Q(X) over the integers and applying formulas to each irreducible factor of degree less than 5. For factors of higher degree only those of the form aX**n+b will be solved. This facility can be augmented by including more special cases for polynomials of higher degree. If the integrand is not rational, the problem of finding poles can be much more difficult, although SOLVE can find some of these also. In any event, the method of residues can be successful only if all relevant poles can be located.

Let p be a root of Q(X) = 0 of multiplicity m which is obtained employing SOLVE. It is not necessarily true that p is a pole of R(Z) of order m. This is because the rational function package [10] in MACSYMA does GCD cancellations only over the integers while R(Z) is a rational function over the complex numbers. Therefore p may not be a pole of R(Z) or p may be a pole of R(Z) of order less than m. Fortunately WANDERER can pretend that p is really a pole of order m and proceed with the algorithm for computing residues which will produce a 0, if p is not a pole, and the correct residue, in case p is a pole of order less than m. More discussion on the computation of residues can be found in Sect. 6-1.

Thus, as the reader can easily verify, (D2) of example (C2), in page 59, is valid even if A=0 and B=1 (or A=0 and B=9). As another example, correct answers were obtained in the following integrals even if %I, the square root of -1, is treated as a simple pole of the integrand.

(D6) $$\frac{X - \%I}{(X - 2\ \%I)(X^2 + 1)}$$

(C7) DEFINT(D6,X,MINF,INF)

(D7) $$\frac{2\ \%PI}{3}$$

(C8) DEFINT(D6,X,0,INF)

(D8) $$\frac{\%I\ LOG(2) + \%PI}{3}$$

A special case check is provided as an auxiliary to the above algorithm.

$$\int_0^\infty \frac{X^{P-1}\ dX}{(A\ X^N + B)^M}$$

$$= (B/A)^{P/N} \binom{- P/N + M - 1}{M - 1} \frac{\%PI}{B^M\ N\ SIN(\%PI\ P/N)}$$

for M, N, P positive integers, M > P/N, P not divisible by N and AB > 0.

For the case M=1, AB<0, the following Cauchy principal value is used.

$$-\left(-\frac{A}{B}\right)^{1 - P/N} \left(\frac{\%PI}{A\,N}\right) COT\left(\frac{\%PI\ P}{N}\right)$$

This special case is not necessary for the algorithm but it is very helpful as far as program efficiency is concerned, especially for large M or N.

Sometimes substantial computing time can be saved by application of differentiation techniques in integration problems. Consider the infinite integral

$$(5) \qquad J = \int_{-\infty}^{\infty} \frac{dX}{(X^2 + X + K)^{10}}.$$

If the algorithm for rational functions were used straightforwardly, differentiation would have to be carried out 18 times to obtain the sum of residues in case both poles lie above the real axis. But since

$$J = -\frac{1}{9!}\left(\frac{d}{dK}\right)^9 \int_{-\infty}^{\infty} \frac{dX}{X^2 + X + K},$$

it is only necessary to differentiate a quadratic expression once to obtain the sum of residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dX}{X^2 + X + K}.$$

After this is obtained, 9 more differentiations with respect to a parameter K are needed. It means a saving of at least 8 differentiations.

In fact, whenever the denominator Q(X) of the integrand is of the form

v(X)**n

with v(X) a polynomial and n ≥ 2, this method of differentiation may be applicable if the degree of Q(X) is large compared to that of the numerator P(X).

We have included this technique in WANDERER. Let

n = deg(P(X)),

s = deg(v(X)),

Q(X) = v(X)**m.

Here is a brief description of the algorithm.

If 2 ≥ m*s-n, this algorithm is not applicable, otherwise (m*s-n > 2)

(i) Set r to the least integer ≥ (n+2)/s

(ii) If m > r

(a) Compute the original integral with m replaced by r and v(X) replaced by v(X) + ZP where ZP is a new parameter introduced into the problem. Let the answer thus obtained be ANS(ZP).

(b) Return the following as the answer

$$(-1)^{m-r} \frac{(r-1)!}{(m-1)!} \left(\frac{d}{dZP}\right)^{m-r} ANS(ZP) \Bigg|_{ZP=0}$$

(iii) otherwise (m=r), the algorithm is not necessary and thus will not be applied.

The parameter ZP used in this algorithm makes the symbolic differentiation in step (ii-b) possible at all times. The final answer is obtained by setting ZP to zero. Such a parameter will be used again later and will be referred to as the zero parameter.

For example,

```
                                 1
(D9)                      -------------
                            2         3
                          (X  + X + 1)

(C10)DEFINT(D9,X,0,INF)@
                             4 %PI     1
                           --------- - -
                           9 SQRT(3)   2
```

In this example, we have n=0, m=3 and s=2. Thus r=1 and the value of

$$\int_0^\infty \frac{dX}{(X^2 + X + 1)}$$

is computed first.

2. Infinite Integrals Involving Algebraic Irrational Functions

2.1 Evaluation by Residues

Let R(X) be a rational function. Consider

$$D = \int_0^\infty X^{**}K^*R(X)\ dX, \qquad -1 < K < 1$$

If

$$\text{LIMIT}(X^{**}(K+1)\ R(X),\ X,\ 0,\ \text{PLUS}) = 0,$$

$$\text{LIMIT}(X^{**}(K+1)\ R(X),\ X,\ \text{INF}) = 0,$$

and R(X) has no poles of order greater than one on the positive real axis, then D can be evaluated by applying residue theory to the contour integral

$$\int_C f(Z)\ dZ, \qquad f(Z) = (-Z)^{**}K\ R(Z)$$

around the contour C shown in fig. 2. The limit conditions given above are convergence tests for D. They are computed by DELIMITER. In computing these integrals, the integration program will first obtain two sums of residues S1 and S2 of the function f(Z) by executing the following two steps.

> If R(Z) has poles off the positive real axis, set S1 to $\sum$Res(f(Z)) at these poles, otherwise set S1 to

0.

If R(Z) has simple positive poles, set S2 to $\sum$Res(f(Z)) at these poles, otherwise set S2 to 0.

Now the answer to the integral D is given by

```
      %PI S1
---------------- - %PI COT(%PI (K + 1)) S2
SIN(%PI (K + 1))
```

For example, MACSYMA produced the following result.

```
(C1) 1/((1+X)*X**(1/2))@
                                    1
(D1)                         ---------------
                             SQRT(X) (X + 1)
(C2) DEFINT(%,X,0,INF)@
(D2)                               %PI
```

[3]

2.2 Integrals Related to the Beta Function

From the definition of beta function

$$\text{BETA(K,S)} = \int_0^1 \text{X**(K-1) (1-X)**(S-1) dX,}$$

where K > 0 and S > 0, One can deduce the following relation

$$\int_0^{\infty} \frac{X^{K-1}}{(CX+D)^S}\, dX = \frac{D^{K-S}\ \mathrm{BETA}(K, S-K)}{C^K}$$

where CD > 0 and S > K > 0.

By a further change of variable in this formula, the following very useful relation is obtained.

(a) $$\int_0^{\infty} \frac{X^{K-1}}{(CX^r+D)^S}\, dX = \frac{\mathrm{BETA}(A,B)}{C^A\ D^B\ r}, \qquad r > 0$$

where A = K/r > 0, and B = S-A > 0.

This rather general formula covers many interesting infinite integrals involving algebraic irrational functions. This is built into the programs by using special purpose pattern recognition routines (see chapter 6) to examine whether the integrand is of the particular form (a). A simplification routine for Beta function was also needed to put the results in a simpler and better looking form.

(D3) $$\frac{X^K}{X + 3}$$

(C4) DEFINT(D3,X,0,INF)@ [12]

```
IS THE EXPRESSION
K + 1
POSITIVE, NEGATIVE, OR ZERO
```

POSITIVE@

IS THE EXPRESSION
$- K$
POSITIVE, NEGATIVE, OR ZERO

POSITIVE@

(D4) $$3^{K} \mathrm{BETA}(K + 1, - K)$$

2.3 Evaluation by Trigonometric Substitution

If R(X,Y) is a rational functions in the two variables X and Y, then an integral of the form

$$U = \int_0^{\infty} R(X, \mathrm{SQRT}(X{*}{*}2-A{*}{*}2))\, dX$$

can be transformed to an infinite integral of a rational function which has been discussed in Sect. 4-1. The transformation below is easy to verify.

Let

$$Y = \frac{\mathrm{SQRT}(X^{2} - A^{2})}{X + A}.$$

U becomes

$$4\, A \int_0^{1} R\left(\frac{A\,(Y^{2} + 1)}{1 - Y^{2}}, \frac{2\, A\, Y}{1 - Y^{2}}\right) \frac{Y\, dY}{(1 - Y^{2})^{2}}.$$

Now let $Z = Y/(1 - Y)$ in the above integral, it can be shown that U is equivalent to

$$\text{(I)} \quad 4\,A \int_0^{\infty} R\left(\frac{A\,(2\,Z^2 + 2\,Z + 1)}{2\,Z + 1}, \frac{2\,A\,Z\,(Z + 1)}{2\,Z + 1}\right) \frac{Z\,(Z + 1)\,dZ}{(2\,Z + 1)^2}.$$

Although the above is all that is needed for the given integral U, it is sometimes more efficient to transform U into a finite integral of a rational function of the trigonometric functions. That is, by setting COS(t) = A/X in U, one may deduce that

$$\text{(II)} \qquad U = A \int_0^{\pi/2} R\left(\frac{A}{\mathrm{COS}(t)}, A\,\mathrm{TAN}(t)\right) \left(\frac{\mathrm{SIN}(t)}{\mathrm{COS}^2(t)}\right) dt$$

Both of these methods (I) and (II) have been included in the programs. Our heuristic rule for applying these transformations is as follows.

> Apply (II) if it transforms the given integrand into the form
>
> COS(t)**m SIN(t)**n,
>
> otherwise apply (I).

For example,

(D5) $$\frac{1}{X^2 \, SQRT(X^2 - A^2)}$$

(C6)DEFINT(D5,X,0,INF)@

(D6) $$\frac{1}{A^2}$$

2.4 Differentiation with Respect to a Parameter

We have included two formulas in WANDERER for infinite integrals involving algebraic irrational functions. They are

$$\int_0^\infty \frac{dX}{(AX^2 + B X + C)^{3/2}} = \frac{1}{SQRT(C)\ (B/2+SQRT(AC))} = U(A,B,C),$$

with $A \geq 0$, $C > 0$, and $B > -SQRT(AC)$, and

$$\int_0^\infty \frac{X \ dX}{(AX^2 + B X + C)^{3/2}} = \frac{1}{SQRT(A)\ (B/2+SQRT(AC))} = V(A,B,C)$$

where $A > 0$, $C \geq 0$ and $B > - SQRT(AC)$.

They are included because any integral of the form

(iii)
$$\int_0^{\infty} \frac{X^M \, dX}{(AX^2 + B X + C)^{N+3/2}}$$

can be computed from these known results. Our tool in doing this is differentiation with respect to an appropriate parameter. Let Z1, Z2 and Z3 be three zero parameters and

$$H = (-1)^N \frac{\text{SQRT}(\%PI)/2}{\text{GAMMA}(\frac{3}{2} + N)}.$$

The algorithm is as follows.

Upon deciding that the integrand is as given in (iii), WANDERER has obtained the values of M, N, A, B and C. The answer to the given integral is then computed using the simple procedure:

If M = 0 and N = 0, return U,

If M = 1 and N = 0, return V,

If 2N+2 $\leq$ M, the given integral is divergent.

If N $\geq$ M, return

$$H \left\{ \frac{d^N}{dZ3^{N-M} \, dZ2^M} \; U(A,B+Z2,C+Z3) \right\},$$

If M = N+1, return

$$H \left(\frac{d}{dZ2}\right)^N V(A,B+Z2,C),$$

If 2(N+1) > M > N+1, then

(i) if M is even set r = M/2, return the answer

$$H \left\{ \frac{d^N}{dZ3^{N-r}\, dZ1^r} U(A+Z1,B,C+Z3) \right\},$$

(ii) if M is odd, set r = (M-1)/2 and return

$$H \left\{ \frac{d^N}{dZ3^{N-r}\, dZ1^r} V(A+Z1,B,C+Z3) \right\}.$$

A simple example is

(D7) $$\frac{X^6}{(X^2 + X + 1)^{\frac{15}{2}}}$$

(C8) DEFINT(D7,X,0,INF)@

(D8) $$\frac{1024}{6567561}$$

3. Infinite Integrals Involving Trigonometric Functions.

3.1 Integrals from Minus Infinity to Infinity

Let R(X) be a rational function which has no real poles and

LIMIT(R(X),X,INF) = 0.

The integral

$$L = \int_{-\infty}^{\infty} \%E^{(\%I\ m\ X)}\ R(X)\ dX, \qquad m > 0$$

is convergent. Its value can be obtained by evaluating the contour integral

$$J = \int_{C} \%E^{(\%I\ m\ Z)}\ R(Z)\ dZ$$

around the contour C given in fig. 1.

Let Cr be a circular arc with center at Z = 0, radius r and argument t, t1 < t < t2. Jordan's Lemma [7] shows that if f(Z) approaches 0 uniformly on Cr as r approaches INF,

$$\underset{r \to +\infty}{\mathrm{LIMIT}} \int_{C_r} \%E^{(\%I\ m\ Z)}\ f(Z)\ dZ=0, \qquad \text{for } t1 \geq 0,\ t2 \leq \%PI$$

and that

$$\lim_{r \to +\infty} \int_{C_r} \%E^{(-\%I\ m\ Z)}\ f(Z)\ dZ = 0, \quad \text{for } t1 \geq \%PI,\ t2 \leq 2\%PI.$$

Therefore L = J as r tends to infinity and the method of residues can be applied. Moreover, for m < 0 one may use the same method by using a contour similar to C in the lower half complex Z-plane.

For real m, n and p, let T(X) be SIN(mX), COS(nX), %E**(%I p X) or a function involving sums and/or products of these functions. In complex exponential form T(X) is a sum of constant multiples of functions of the form %E**(%I k X), k real.

Therefore

$$\int_{-\infty}^{\infty} T(X)\ R(X)\ dX$$

can be integrated using complex contour integral. In doing such a problem, terms in the integrand are sorted into two parts. One part requires a contour in the upper-half Z-plane, the other a contour in the lower-half.

Before discussion of additional methods, let us see a few examples computed using methods discussed so far.

```
                                COS(X)
(D1)                            ------
                                  2
                                 X  + 1

(C2) DEFINT(D1,X,MINF,INF)@                                     [12]

                                %PI
(D2)                            ---
                                %E

(C3) X*SIN(X)/(X**2+1)@

                               X SIN(X)
(D3)                           --------
                                 2
                                X  + 1

(C4) DEFINT(D3,X,MINF,INF)@                                     [12]

                                %PI
(D4)                            ---
                                %E

(C5)X*COS(X)/(X**2+1)@

                               X COS(X)
(D5)                           --------
                                 2
                                X  + 1

(C6) DEFINT(D5,X,MINF,INF)@

(D6)                              0

(C7) X*COS(X)/(X**2+X+1)@

                               X COS(X)
(D7)                          ----------
                                2
                               X  + X + 1

(C8) DEFINT(D7,X,MINF,INF)@
```

```
(D8)

                    1                  (SQRT(3) - %I)/2
(%PI/2) ((- ------- - %I) %E
             SQRT(3)

                                 1         (- SQRT(3)- %I)/2
                    + (%I - -------) %E                      )
                             SQRT(3)

(C9) 1/(%E**(%I*X)*(X**2+1))@

                                            1
(D9)                               ---------------
                                     %I X    2
                                   %E     (X  + 1)

(C10) DEFINT (D9,X,MINF,INF)@

                                          %PI
(D10)                                     ---
                                          %E

(C11) SIN(X)*D9@

                                        SIN(X)
(D11)                              ---------------
                                     %I X    2
                                   %E     (X  + 1)

(C12) DEFINT(D11,X,MINF,INF)P

                                          %PI
(D12)                                     ---
                                          %E

(C13) D11*(%E**(-%I*X))
                                        SIN(X)
(D13)                             -----------------
                                    2 %I X    2
                                  %E       (X  + 1)

(C14) DEFINT(D13,X,MINF,INF)@
```

(D14)

$$\%E^{\frac{\%PI}{2}}$$

3.2 Integrals from 0 to Infinity

Let us now discuss some integrals involving trigonometric functions over the range (0 INF). First consider the integrals

$$I1 = \int_0^{\infty} COS(k\ X^n)\ dX$$

and

$$I2 = \int_0^{\infty} SIN(k\ X^n)\ dX$$

where k a nonzero real constant and n > 1.

Here again, the residue theory can be applied to evaluate the integrals. This time the shape of the contour is slightly different. The contour is a sector of a circle consisting of a portion of a circular arc with center at the origin and two straight lines joining the end points to the origin. (fig. 3) The sector angle depends on n and is %PI/(2n).

Without loss of generality, let us assume that k > 0.

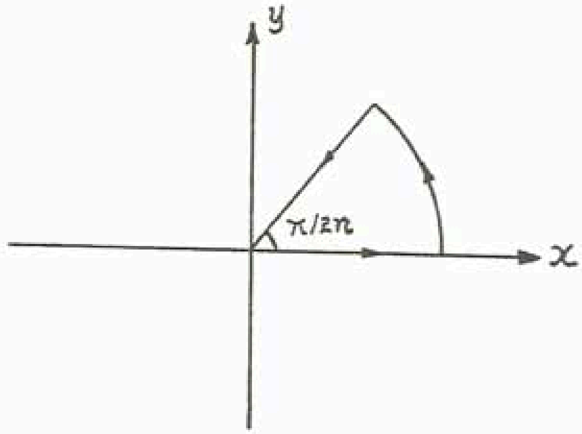


fig.3

By Cauchy's integral theorem,

$$\int_C \%E^{(\%I\ k\ Z^n)}\, dZ = 0.$$

Since contribution from the circular arc vanishes as r approaches INF, it can be shown that

$$\int_0^\infty \%E^{(\%I\ k\ Z^n)}\, dZ$$

$$= \%E^{\%I\ \%PI/(2\ n)} \int_0^\infty \%E^{-k\ r^n}\, dr$$

The integral in the right hand side of this equation can be expressed in terms of GAMMA function (see Sect. 5). Let us assume that

$$\int_0^\infty \%E^{-k\ r^n}\, dr = G$$

Therefore, taking real and imaginary parts of the above relation, we have

I1 = COS(%PI/2n) G

and I2 = SIN(%PI/2n) G.

For example

```
                                    7/3
(D15)                         COS(9 X   )
(C16) DEFINT(D15,X,0,INF)@

                           3       3 %PI
                    3 GAMMA(-) COS(-----)
                           7        14
(D16)               ---------------------
                               3/7
                            7*9

(C17) SIN(9*X**(7/3))@

                                    7/3
(D17)                         SIN(9 X   )
(C18) DEFINT(D17,X,0,INF)@

                           3       3 %PI
                    3 GAMMA(-) SIN(-----)
                           7        14
(D18)               ---------------------
                               3/7
                            7*9
```

Incidentally, if $n \leq 1$, I1 and I2 do not converge.

A more general integral than I1 and I2 is

$$J = \int_0^{\infty} X^m \ \mathrm{EXP}(\%I \ k \ X^n) \ dX$$

where $n > 0$, $\mathrm{Rl}(m) > -1$, k real and nonzero and $n - \mathrm{Rl}(m) > 1$.

We shall consider the case $k < 0$. The case $k > 0$ is entirely analogous. Let us take a sector-shaped contour in the fourth quadrant as shown in fig. 4.

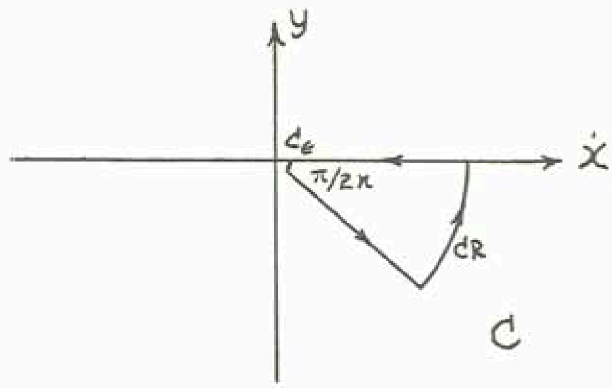


fig. 4

By Cauchy integral theorem,

$$\int_C Z^m \ \%E^{(\%I \ k \ Z^n)} \ dZ = 0$$

which implies that

$$J = \mathrm{EXP}(-\%PI \ \%I(m+1)/(2n)) \int_0^{\infty} R^m \ \mathrm{EXP}(k \ R^n) \ dR$$

$$= EXP(-\%PI\ \%I(m+1)/(2n))\ GAMMA\left(\frac{s}{n(-k)^{s}}\right)$$

where s = (m+1)/n.

Let us look at an example.

(D19)
$$\frac{X^{(2\ \%I\ +\ 3)}}{\%E^{(\%I\ X^{3}\ /\ 2)}}$$

(C20) DEFINT(D19,X,0,INF)@

(D20)
$$\frac{GAMMA\left(\frac{2\ \%I\ +\ 4}{3}\right)}{3\ (\%I\ /\ 2)^{(2\ \%I\ +4)\ /\ 3}}$$

A similar result for k > 0 is given by

$$J = EXP(\%PI\ \%I(m+1)/(2n))\ GAMMA\left(\frac{(m+1)/n}{k^{(m+1)/n}\ n}\right).$$

The following integrals can be obtained readily from the above relation.

$$\int_0^{\infty} COS(k\ X^{n})\ X^{m}\ dX \quad \text{and} \quad \int_0^{\infty} SIN(k\ X^{n})\ X^{m}\ dX$$

For instance,

```
(C21) SIN(P*X)/(X**(1/2))@

                                   SIN(P X)
(D21)                              --------
                                   SQRT(X)

(C22) DEFINT(D21,X,0,INF)@                                   [3]

                                 SQRT(%PI)
(D22)                         ----------------
                              SQRT(2) SQRT(P)
```

Many of the foregoing results depend on the verification of the fact that contributions from certain parts of a contour vanish after taking a limit. We have omitted these proofs. A typical such proof is to show

$$\int_{CR} f(Z)\ dZ = \int_{C_\epsilon} f(Z)\ dZ = 0.$$

The above derivation has relied on this fact. The proof is in appendix B.

Another interesting integral is

$$U = \int_0^\infty \mathrm{SIN}^K(X)\ X^{-N}\ dX$$

where N and K are positive integers, K   N > 2 and (K + N) even. The integral U can be evaluated by use of the recurrence relation

$$\int_0^{\infty} \mathrm{SIN}^r(X)\, X^{-s}\, dX = \frac{r\,(r-1)}{(s-1)(s-2)} \int_0^{\infty} \mathrm{SIN}^{r-2}(X)\, X^{-(s-2)}\, dX$$

$$+ \frac{r^2}{(s-1)(s-2)} \int_0^{\infty} \mathrm{SIN}^r(X)\, X^{-(s-2)}\, dX,$$

where $r > (s-1) > 1$. As one can see, repeated application of this relation will reduce U to a sum of integrals of the same form as U but with N = 1 or 2. For N=1, AUDITOR uses the following formula

$$\int_0^{\infty} \mathrm{SIN}^p(X)\, X^{-1}\, dX = \int_0^{\pi/2} \mathrm{SIN}^{p-1}(X)\, dX, \qquad p > 0 \text{ odd.}$$

The integral on the right hand side can be evaluated easily. Integrals of this type are considered in Sect. 5-2.2. For N=2, we have

$$\int_0^{\infty} \mathrm{SIN}^q(X)\, X^{-2}\, dX = \frac{\%PI}{2} \binom{q-3/2}{q-1}$$

where q is an integer $\geqq 2$. For example

(D1)
$$\frac{\mathrm{SIN}(R\ X)}{X}$$

```
(C2) DEFINT(%,X,0,INF)@
                                         %PI
(D2)                                     ---
                                          2

(C3) SIN(Q*X)**2/(X**2)@
                                        2
                                     SIN (Q X)
(D3)                                 ---------
                                          2
                                         X

(C4) DEFINT(%,X,0,INF)@
                                       %PI Q
(D4)                                   -----
                                         2
```

4. Infinite Integral Involving Logarithm Functions

4.1 Evaluation by Recursion and Contour Integration

Let R(X) be a rational function which is even in X. The integral

$$V(N) = \int_0^{\infty} LOG^N(X)\ R(X)\ dX, \quad R(X) = P(X)/Q(X),$$

with $\deg(Q(X)) - \deg(P(X)) \geq 2$, can be evaluated by applying residue theory in a recursive manner. Recall that LOG is an abbreviation of PLOG when the argument is real and positive. Consider the contour integral

$$J(N) = \int_C F(Z)\ dZ, \qquad F(Z) = PLOG^N(Z)\ R(Z)$$

where C is the indented contour in fig. 5. As r approaches

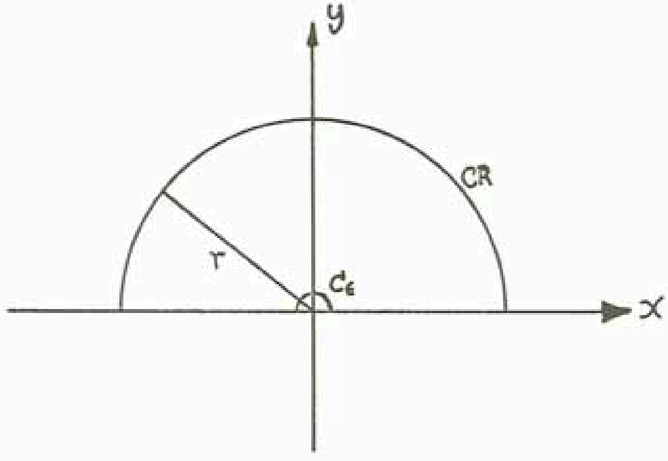


fig. 5

INF and ϵ approaches 0, contributions from CR and C_ϵ vanish. Therefore

$$J(N) = \int_0^\infty R(t)\ LOG^N (\%E^{\%PI\ \%I}\ t)\ dt + V(N)$$

$$= \int_0^\infty R(t)\ (LOG(t) + \%PI\ \%I)^N\ dt + V(N)$$

$$= \int_0^\infty R(t) \sum_{K=1}^{N} \binom{N}{K} \%PI^K\ \%I^K\ LOG^{N-K}(t)\ dt + 2\ V(N)$$

where $\binom{N}{K}$ is a binomial coefficient. Thus,

for $N \geq 0$,

$$V(N) = \frac{J(N)}{2} - \frac{1}{2} \sum_{K=1}^{N} \binom{N}{K} \%PI^K\ \%I^K\ V(N - K)$$

where J is obtained by residues. Therefore the result obtained here is

$$V(N) = \%PI\ \%I(\textstyle\sum Res(F) \text{ over poles of R inside C})$$

$$- \frac{1}{2} \sum_{K=1}^{N} \binom{N}{K} \%PI^K\ \%I^K\ V(N - K).$$

Based on this recurrence relation, V(N) can be computed. Such an algorithm has been included in the programs. The following is an outline of this algorithm.

1) Declare each of AV and AJ to be an 1 X N array
2) Set I to 0 then go to step 4
3) Set AJ(I) to J(I)/2
4) If I = N, return the answer V(N)
5) Set AV(I) to V(I)
6) Set I to I+1 then go to step 3

Computation of V(I):

If I = 0, return the value of the integral

$$\int_0^\infty R(X)\,dX,$$

(storing away information about poles and corresponding residues of R(Z) for computation of J), otherwise, compute from the formula

$$V(I) = AJ(I) - \frac{1}{2}\sum_{K=1}^{I}\binom{I}{K}\ \%PI^{K}\ \%I^{K}\ AV(I-K).$$

Computation of J(I)/2 :

Using existing information of poles and residues of R(Z) compute by residue theory from the formula

$\%PI\ \%I(\sum Res(F)$ over poles of R inside C).

For example,

(D1) $$\frac{\mathrm{LOG}^2(X)}{X^2 + 1}$$

(C2) DEFINT(D1,X,0,INF)@ [1]

(D2) $$\frac{\%PI^3}{8}$$

4.2 Method of Differentiating and Introducing Parameters

A very useful method for integrands involving LOG(X) is differentiation. Consider an integral of the form

$$A(K) = \int_0^{\infty} R(X)\, X^K\, \mathrm{LOG}(X)\, dX, \qquad K \neq 0 \text{ and } -1 < K < 1,$$

where R(X) is rational in X and k is a parameter which occurs nowhere else in the integrand. If the integral

$$B(K) = \int_0^{\infty} R(X)\, X^K\, dX$$

is convergent and can be evaluated, then A(K) is given by

$$A(K) = \frac{d}{dk} B(K).$$

This method of differentiation is valid because B(K) is convergent and A(K) is uniformly convergent for every closed interval contained in the set of points $(K \mid K \neq 0 \text{ and } -1 < K < 1)$.

For example

(D3) $$\frac{X^{K} \; LOG(X)}{X + 3}$$

(C4) DEFINT(D3,X,0,INF)@ [12]

(D4) $$LOG(3)\; 3^{K}\; BETA(K+1,- K) + 3^{K} \left(\frac{D}{Dr} BETA(r, - K)\Big|_{r=(1+K)} - \frac{D}{Dr} BETA(K + 1,r)\Big|_{r=-K} \right)$$

In the foregoing discussion, K has been assumed symbolic. Yet, in solving actual problems K may very well be a number. This difficulty can be overcome by introducing a zero parameter ZP. We first replace X**k by X**(k+ZP) in f(X). This permits us to proceed as above and then differentiate with respect to ZP. After differentiation the result is then evaluated at ZP = 0.

The property of the logarithm function

$$-LOG(X) = LOG(1/X)$$

can sometimes be used in evaluating integrals of the form

$$U = \int_0^\infty f(X)\ LOG(X)\ dX.$$

In fact, if a change of variable Y = 1/X is made in U and if the new integral happens to be

$$\int_\infty^0 f(Y) LOG(Y)\ dY,$$

then the value of the given integral U is 0. For example

$$\text{(D5)} \qquad \frac{(ARCTAN(X^{1/3}) + ARCTAN(X^{-1/3}))\ LOG(X)}{X^2 + 1}$$

```
(C6) DEFINT(D5,X,0,INF)@
(D6)                          0
```

4.3 Integration by Parts

Another nice property of the logarithm functions is that their derivatives are often simpler than the function themselves. Because of this, integration by parts is frequently a suitable method for integrals involving them.

If the indefinite integral

$$U(X) = \int f(X)\ dX$$

can be obtained and the integral

$$J = \int_a^b LOG(g(X))\ f(X)\ dX$$

is convergent, then

$$J = LOG(g(X))\ U(X)\Big|_a^b - \int_a^b \left(\frac{U(X)g'(X)\ dX}{g(X)}\right).$$

The SIN integration program [20] in MACSYMA is used to compute the indefinite integral U(X). WANDERER uses this method to evaluate integrals of the form

$$\int_0^\infty X^{-L}\ LOG(1 + X^a)\ dX, \qquad a+1 > L > 1.$$

Here is an integral evaluated by the method of integration by parts.

```
                          7/2
                     LOG(X    + 1)
(D7)                 -------------
                           2
                          X

(C8) DEFINT(D7,X,0,INF)@

                          %PI
(D8)                  ----------
                          5 %PI
                      SIN(-----)
                            7
```

5. Infinite Integrals Involving Exponential Functions

5.1 Method of Substitution

Integrals with integrands which are rational functions of %E**(K*X), K real and non-zero, are relatively easy to compute. Without loss of generality, let us assume K > 0 and consider

$$(1) \qquad \int_{-\infty}^{\infty} R(\%E**(K*X))\ dX.$$

If one makes a change of variable

$$(2) \qquad Y = \%E**(K*X),$$

then (1) becomes

$$(1/K)\int_{0}^{\infty} R(Y)/Y\ dY.$$

This integral converges if (1) does. That is to say R(Y) has Y as a factor if (1) converges. For the integral

$$(3) \qquad \int_{0}^{\infty} R(\%E**(K*X))\ dX$$

one can make a similar change of variable

$$(4) \qquad S + 1 = \%E**(K*X)$$

which gives

(5) $$\int_0^{\infty} R(S+1)/(S+1)\ dS.$$

Integrals of the form of (3) and (5) can be evaluated by contour integral and residue theory as indicated in sect. 1. The methods (2) and (4) can in general be applied to any integral of the form

$$\int_a^b f(g(X))\ dX$$

and the resulting integral may be much simpler. For instance

```
                         1
(D1)           -----------------
                X/3    5
              %E    (----- + 7)
                        X/3
                      %E

(C2) DEFINT(D1,X,0,INF)@
                            7
                     3 LOG(--)
                           12
(D2)               - ---------
                         5

                      X/4
                    %E
(D3)             -----------
                      X/2
                 9 %E    + 4

(C4) DEFINT(D3,X,MINF,INF)@
```

(D4) $$\frac{\%PI}{3}$$

5.2 Use of Contour Integration

A somewhat more interesting integral is

$$I = \int_{-\infty}^{\infty} R(\%E**X)\ P(X)\ dX$$

where P(X) is a polynomial and R(X) a rational function with complex coefficients such that

LIMIT(R(%E**X),X,INF) = 0

and

LIMIT(R(%E**X),X,MINF) = 0.

Let us first determine a polynomial Q(Z) with complex coefficients such that

(6) Q(X) - Q(X + 2 %PI %I) = P(X).

Q(Z) exists and can be computed by the method of undetermined coefficients. Now consider a contour integral

$$J = \int_{C} R(\%E**Z)\ Q(Z)\ dZ$$

taken around a rectangular contour as shown in fig. 6.

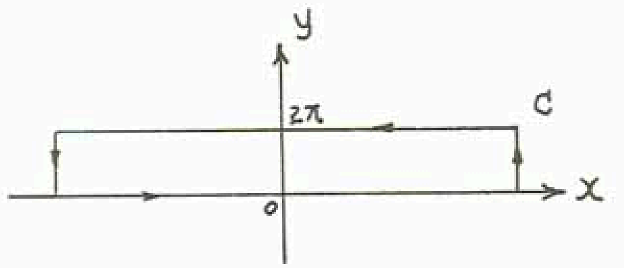


fig. 6

As the absolute value of X approaches infinity, contributions from the vertical segments of the rectangular contour vanish. Hence,

$$J = \int_{-\infty}^{\infty} R(\%E**X)\ Q(X)\ dX - \int_{-\infty}^{\infty} R(\%E**X)\ Q(X+2\ \%PI\ \%I)\ dX.$$

It follows from (6) that I=J. Now the problem of evaluating the integral I has been reduced to finding the residues of R(%E**Z)Q(Z) for $0 \leq$ Im(Z) < 2 %PI. To do this, the poles of R(Z) are obtained first. If w is such a pole then

GLOG(w)

where GLOG stands for the branch of LOG with imaginary part between 0 and 2*%PI, is a pole of the same order for R(%E**Z)*Q(Z) inside the closed contour. All such poles can be obtained in this manner. One example is

```
(D5)                           X
                         ------------
                         SINH(X) - %I

(C6) DEFINT(D5,X,MINF,INF)@
```

(D6) %PI

Note that the integrand (D5) has a pole of order two at $Z = \%PI\ \%I/2$.

5.3 The GAMMA Function and Related Integrals

A very important function closely related to the evaluation of infinite integrals involving exponentials is the GAMMA function generally defined as

$$\mathrm{GAMMA}(Z) = \int_0^{\infty} \%E{**}(-t)\ t{**}(Z-1)\ dt, \qquad \mathrm{Rl}(Z) > 0.$$

Also of use is its logarithmic derivative, the PSI function

$$\mathrm{PSI}(Z) = (\frac{d}{dZ}\ \mathrm{GAMMA}(Z))/\mathrm{GAMMA}(Z).$$

A simplification routine for GAMMA function has been written to make use of the many properties of this function.

From the definition of the GAMMA function, one can derive the following very useful relation

$$\int_0^{\infty} X^{D}\ \%E^{C - A X^{B}}\ dX = \frac{\mathrm{GAMMA}(a)\ \%E^{C}}{A^{a}\ B},$$

where a = (D + 1)/B, RI(A) > 0, RI(D) > -1 and RI(B) nonzero.

WANDERER has programs designed to recognize this form and return the result. Of course these programs have to examine the signs of the relevant quantities carefully before generating an answer.

For example:

```
                                       2
                                     -X
(D7)                               %E
(C8) DEFINT(D7,X,0,INF)@
                                   SQRT(%PI)
(D8)                               ---------
                                       2
```

5.4 Integral Related to the Laplace Transform

Let f(t) be a function of a real variable t, then its Laplace transform L(f(t)) is defined as

$$L(f(t)) = F(s) = \int_0^{\infty} \%E^{(-t\ s)}\ f(t)\ dt.$$

Many of such integrals can be evaluated by the programs. Here are some examples produced by WANDERER.

```
(D9)                         1
                       -------------
                                 S T
                       SQRT(T) %E

(C10) DEFINT(D9,T,0,INF)@

IS THE EXPRESSION
  S
POSITIVE, NEGATIVE, OR ZERO

POSITIVE@

                          SQRT(%PI)
(D10)                     ---------
                           SQRT(S)


(C11) SIN(S*X)/(%E**X)@


                           SIN(S X)
(D11)                      --------
                               X
                             %E

(C12)DEFINT(D11,X,0,INF)@

                              S
(D12)                      ------
                            2
                           S  + 1
```

When f(t) involves trigonometrical or hyperbolic functions, they are expanded into exponentials before L(f(t)) is computed. When LOG(t) is a factor of f(t) the method of differentiation with respect to a parameter can be applied if f(t) also has X**k as a factor, Rl(k) > 0. In this case a new function g(t,ZP) is constructed by replacing LOG(t) with t**ZP where ZP is a zero parameter that has been specially introduced. The Laplace transform of $g(t,ZP)$ will be computed first. Suppose G in

$$G(s,ZP) = L(g(t,ZP)) = \int_0^{\infty} g(t,ZP)\ \%E^{(-t\cdot s)}\ dt$$

has been obtained. Now, as mentioned before in sect. 4, all that needs to be done to obtain $L(f(t))$ is to compute dG/dZP. That is

$$L(f(t)) = dG/dZP \text{ at } ZP = 0.$$

This is true for

$$f(t) = dG/dZP \quad \text{at } ZP=0,$$

and the fact that there exists some sufficiently small closed interval containing 0 in which the following integral is uniformly convergent:

$$\int_0^{\infty} \%E^{(-t\ s)} \frac{dG}{d\ ZP}\ dt.$$

For example,

```
                               -S T  1/3
(D13)                        %E     X     LOG(X)

(C14) DEFINT(D13,T,0,INF)@

                    1                4            4/3
(D14)     - GAMMA(-) (LOG(S)- PSI(-) ) /(6 S    )
                    3                3
```

CHAPTER V

INTEGRALS OVER A FINITE RANGE

0. Introduction

Many finite integrals are proper integrals whose indefinite integrals exist in closed form and can be computed rather easily. For such an integral, the evaluation method is very straight-forward. WANDERER simply obtains the corresponding indefinite integrals and then substitutes the limits of integration. The antiderivatives are computed by use of SIN. This method will be referred to as the antiderivative method.

A finite integral

$$\int_a^b f(X)\, dX$$

is improper if f(X) becomes infinite at some point c, $b \geq c \geq a$. In order to avoid having to spend computation time looking for an unknown number of singularities of f between a and b, WANDERER will assume that integrands of finite integrals, with the exception of rational integrands, can become infinite only at the end points of the range of integration. This is not a severe restriction and does not decrease the number of integrals it can handle. This is true in the sense that any given range can be subdivided to conform to the above convention. This convention makes it easy to determine whether a given finite integral is improper. WANDERER simply checks the value of the integrand at the limits of integration. If the given integral is improper, its divergence is tested before any attempt at evaluation. WANDERER uses a limit test for absolute divergence which is discussed in Scet. 6-6. If the given integral diverges, WANDERER will so indicate in the output. If the antiderivative can be computed, then the answer is sometimes obtained by employing a limiting process when substituting the upper and lower limits of integration.

In this chapter, attention will be focused on the definite integrals whose corresponding indefinite integrals are difficult to compute or do not exist.

1. Finite Integrals of Rational Functions.

For integrals such as

$$U = \int_a^b R(X)\, dX$$

where R(X) is rational in X, a and b are finite, WANDERER computes U by transforming it into an infinite integral by a change of variable.

Let us write

(1) $$Y = \frac{(X - a)}{(b - X)}, \qquad X = \frac{(a + Y)}{(1 + Y)},$$

$$dX = \frac{(b - a)\, dY}{(1 + Y)^2}$$

then it is evident that

$$U = (b - a)\int_0^{\infty} R\left(\frac{b\,Y + a}{Y + 1}, \frac{dY}{(Y + 1)^2}\right).$$

This integral can be integrated readily by methods of contour integration and other means discussed in sect. 4-1.

As an example let R(X) be the expression (D1)

(D1) $$\frac{1}{X^2 - 3}$$

The indefinite integral of R(X) from 0 to 1 computed by the substitution method above is given by

(C2) DEFINT(D1,X,0,1)@

$$\text{(D2)} \qquad \frac{\text{LOG}(2 - \text{SQRT}(3))}{2\ \text{SQRT}(3)}$$

One knows that the indefinite integral of (D1) exists. In fact the following has been obtained using the command INTEGRATE in MACSYMA

```
(C3)INTEGRATE(D1,X)@

(D3)  (LOG(2 X - 2 SQRT(3)) - LOG(2 X + 2 SQRT(3)))
                                            /(2 SQRT(3))
```

The reader may easily obtain a result equivalent to (D2) by Substituting in (D3) the limits of integration.

2. Rational Functions of Trigonometric Functions

2.1 A Typical Application of Contour Integration

If R(X,Y) is a rational function in two variables X and Y, an integral in the form

$$U = \int_0^{2\pi} R(\text{COS}(X),\text{SIN}(X))\ dX$$

is easily transformed to an integral around a closed contour. By setting

$$\text{COS}(X) = \frac{Z^2 + 1}{2\ Z}, \qquad \text{SIN}(X) = \frac{Z^2 - 1}{2\ \%I\ Z},$$

and $$dX = \frac{dZ}{\%I\ Z},$$

the given integral U becomes

$$\frac{1}{\%I}\int_{C1} F\left(\frac{Z^2 + 1}{2\ Z}, \frac{Z^2 - 1}{2\ \%I\ Z}\right) \frac{dZ}{Z}$$

where C1 is the positively orientated unit circle with center at Z = 0. This contour integral can then be evaluated by finding the sum of residues inside the circle.

Actually this transformation can be applied in general to any integral in the form

$$\int_0^{2\pi} R(\%E**(\%I\ X))\ dX$$

where R is rational, by the change of variable

$$Z = \%E**(\%I\ X)$$

The transformation process is simple. The key point in this algorithm is the determination of whether the integrand is in fact a rational function of %E**(%I X).

In calculating the sum of the residues, only poles inside the contour contribute. Simple poles on the unit circle, i.e. those with absolute value 1, cause the principal value to be computed. The integral is divergent if poles of order greater than 1 happen to be on the circle.

Examples:

```
(C4) COS(X)**2-SIN(X)@
                    2
(D4)             COS (X) - SIN(X)

(C5) DEFINT(D4,X,0,2*%PI)@

(D5)                       %PI

(C6)  %E**(2*%I*X)/(%E**(%I*X)+3)@

                         2 %I X
                       %E
(D6)                  ----------
                        %I X
                      %E     + 3

(C7) DEFINT(D6,X,0,2*%PI)@

(D7)                     2 %PI
```

2.2 Utilization of the Periodicity of the Trigonometric Functions

In this section integrals of functions involving trigonometric functions over a variety of ranges will be considered.

Let T be a function of X defined by

$$T(X) = \cos^M(X)\,\sin^N(X)$$

where RI(M) > -1 and RI(N) > -1. The following formula can be deduced from the definition of Beta function. (Sect. 4-2.2)

(1) $$\int_0^{\pi/2} T(X)\, dX = \frac{\mathrm{GAMMA}\left(\frac{M + 1}{2}\right)\ \mathrm{GAMMA}\left(\frac{N + 1}{2}\right)}{2\ \mathrm{GAMMA}((N + M + 2)/2)}$$

A simple example is

(D8) $$\mathrm{COS}(X)^{1/3}\ \mathrm{SIN}^{1/2}(X)$$

(C9) DEFINT(D8,X,0,%PI/2)@

(D9) $$\frac{6\ \mathrm{GAMMA}\left(\frac{2}{3}\right)\ \mathrm{GAMMA}\left(\frac{3}{4}\right)}{5\ \mathrm{GAMMA}\left(\frac{5}{12}\right)}.$$

The usefulness of (1) is increased by the fact that it is possible to express definite integrals of T(X) over a variety of ranges in terms of that of T(X) over (0 %PI/2). For instance, the following relations are true for any function f.

$$\int_0^{\pi} f(\mathrm{SIN}(X),\mathrm{COS}(X))\, dX = \int_0^{\pi/2} (f(\mathrm{SIN}(X),\mathrm{COS}(X)) + f(\mathrm{SIN}(X),-\mathrm{COS}(X)))\, dX;$$

$$\int_{0}^{3\pi/2} f(SIN(X),COS(X))\ dX =$$

$$\int_{0}^{\pi} f(SIN(X),COS(X))\ dX + \int_{0}^{\pi/2} f(-SIN(X),-COS(X))\ dX.$$

More generally, let f(X) be a periodic function of X with period 2%PI. That is

$$f(X + 2\%PI) = f(X).$$

An integral of f(X) over some range (a b),

$$S = \int_{a}^{b} f(X)\ dX,$$

can always be written as a sum of integrals in the form

$$(2) \qquad n \int_{0}^{2\pi} f(X)\ dX \ + \int_{0}^{c} f(X)\ dX \ - \int_{0}^{d} f(X)\ dX$$

for some integer n and 2%PI > c, d ≥ 0. This is true for there exist integers p and q such that

$$a = 2\ p\ \%PI + d,$$

and

$$b = 2\ q\ \%PI + c.$$

Then S is equivalent to the sum (2) with n = (q - p).

Programs have been written to perform this reduction and they are applied when the integrand has a period 2%PI and the difference (a - b) has %PI as a factor. some examples computed by WANDERER are

(D10) $\cos(X)^{1/3} \sin^2(X)$

(C11) DEFINT(D10,X,-%PI/2,%PI/2)@

(D11) $$\frac{18\ \mathrm{SQRT}(\%PI)\ \mathrm{GAMMA}(\frac{2}{3})}{7\ \mathrm{GAMMA}(\frac{1}{6})}$$

(C12) COS(X)**3*SIN(X)**2@

(D12) $\cos^3(X) \sin^2(X)$

(C13) DEFINT(D12,X,3*%PI/2,3*%PI)@

(D13) $$\frac{2}{15}$$

3. Finite Integrals of Algebraic Irrational Functions

3.1 Rationalizing the Integrand

If R(X,Y) is a rational function in X and Y, the integral

(1) $$K = \int_b^c R(\mathrm{SQRT}(A^2 - X^2), X)\ dX,$$

where $A \geq c > b \geq -A$, can be rationalized.

Let us write

$$(2) \qquad Y = \frac{A - \mathrm{SQRT}(A^2 - X^2)}{X} = v(X),$$

which gives

$$(3) \qquad X = \frac{2\,A\,Y}{1 + Y^2}$$

Substituting (3) for X, (1) becomes

$$K = 2\,A \int_{v(b)}^{v(c)} R\left(\frac{A(1 - Y^2)}{1 + Y^2}, \frac{2\,A\,Y}{1 + Y^2}\right) \frac{(1-Y^2)\,dY}{(1 + Y^2)^2}$$

which is an integral of a rational function. Integrals of this type have been discussed in Sect. 1.

Similarly for the integral

$$J = \int_b^c R(X, \mathrm{SQRT}(X^2 - A^2))\,dX$$

where $c > b \geq A > 0$, the change of variable

$$t = \frac{\mathrm{SQRT}(X^2 - A^2)}{X + A} = u(X)$$

can be made to convert J to

$$4\,A \int_{u(b)}^{u(c)} R\left(\frac{A\,(t^2 + 1)}{1 - t^2}, \frac{2\,A\,t}{1 - t^2}\right) \frac{t\,dt}{(1 - t^2)^2}$$

which is also an integral of a rational function. Here are two such integrals evaluated by WANDERER.

(D1) $$\frac{1}{X\ \mathrm{SQRT}(X^2 - 9)}$$

(C2) DEFINT(D1,X,3,4)@ [12]

(D2) $$\frac{-\%I}{3}\ \mathrm{PLOG}\left(\frac{3 + \%I\ \mathrm{SQRT}(7)}{4}\right)$$

(C3) 1/((X+1)*(4-X**2)**(1/2))@

(D3) $$\frac{1}{(X + 1)\ \mathrm{SQRT}(4 - X^2)}$$

(C4) DEFINT(D3,X,0,2)@

(D4) $$\frac{\mathrm{LOG}(2 + \mathrm{SQRT}(3))}{\mathrm{SQRT}(3)}$$

Another method is to try to transform the given integral to an infinite integral by the change of variable given in (1) of Sect. 1. For instance the integral

$$\int_A^B R(X, \mathrm{SQRT}((X - A)(B - X)))\ dX$$

can be converted to

$$(B - A)\int_0^{\infty} R\left(\frac{(B + A)\ Y}{Y + 1}, \frac{(B - A)\ Y}{Y + 1}\right)\frac{dY}{(Y + 1)^2}$$

The method of rationalization can also be applied to

$$\int_a^b R(X,\ (CX + D)**(1/Q))\ dX$$

where Q is an integer, C and D are constants.and the range of integration needs not be finite.

For integrals of this type the substitution

$$Y = (C\ X + D)^{1/Q},$$

$$X = \frac{Y^Q - D}{C},$$

$$dX = \frac{Q\ Y^{Q - 1}\ dY}{C},$$

will convert the given integral into that of a rational function in Y.

The principal task in this conversion method is to recognize that the given integrand is of the particular form and to obtain the integer Q. In essence, the algorithm for doing this is

(i) Obtain a list L of all distinct irrational parts in the given integrand.

(ii) If elements in L are fractional powers of an identical linear polynomial in X, the pattern is matched and Q is set to the lcm of the denominators of all the exponents of the linear polynomial, otherwise the pattern is not matched.

3.2 Integrals Related to the BETA Function

The BETA function is defined by the integral

$$\mathrm{BETA}(K,L) = \int_0^1 X^{K-1} (1 - X)^{L-1}\, dX$$

for $\mathrm{Rl}(K) > 0$ and $\mathrm{Rl}(R) > 0$. From this definition one may readily deduce the relation

$$(3) \qquad \int_0^1 X^{K-1} (1 - X^C)^{L-1}\, dX = \frac{1}{C}\, \mathrm{BETA}(\frac{K}{C}, L).$$

WANDERER applies this formula by recognizing the form of the given integrand. In chapter 6 some techniques and programs for pattern recognition are discussed. For integrals in the form

$$\int_A^B (X - A)^{K - 1} (B^C - X^C)^{L - 1} \, dX,$$

a simply substitution,

$$Y = (X - A)/B,$$

will transform it into (3).

4. Finite Integrals Involving Logarithm Functions

When the integrand is a function of LOG(X), a given integral may, in many cases, be evaluated by transforming it into one which involves exponential functions. Consider

$$U = \int_a^b f(LOG(X))\ dX, \qquad b > a \geq 0,$$

the substitution

(I) $$X = \%E^{-Y} \qquad \text{for } 1 > a,$$

or

(II) $$X = \%E^{Y} \qquad \text{for } a \geq 1,$$

converts U to

$$\int_r^s f(-Y)\ \%E^{-Y}\ dY, \qquad r = -LOG(b),\ s = -LOG(a),$$

or

$$\int_r^s f(Y)\ \%E^{Y}\ dY, \qquad r = LOG(a), \quad s = LOG(b),$$

respectively. In case a = 0 and b = 1 (or INF), the use of (I) or (II) will result in an infinite integral which can often be evaluated readily using methods provided in WANDERER. For instance by use of (I) the integral

$$\int_0^1 LOG(X^q)\ dX$$

becomes

$$\int_0^{\infty} (-Y)^q \, \%E^{-Y} \, dY,$$

which is an infinite integral already studied (Sect. 4-5.3). The method (I) or (II) can also be applied to integrals of the form

$$\int_a^b X^r f(\mathrm{LOG}(X)) \, dX,$$

as in

$$\int_0^1 \mathrm{LOG}^S \left(\frac{1}{X}\right) X^R \, dX = \int_0^{\infty} Y^S \, \%E^{-RY-1} \, dY.$$

For integrands involving LOG(f(X)), it may sometimes be possible to simplify the given integral by the substitution

$$Y = f(X), \quad X = g(Y)$$

where the function g is the inverse function of f. A very simple application of this method is conversion of

$$\int_{a^2}^{\infty} X^2 \mathrm{LOG}(\mathrm{SQRT}(X)+a) \, dX$$

to the integral

$$2\int_0^{\infty} (Y-a)^2 \mathrm{LOG}(Y) \, dY.$$

Techniques of differentiation with respect to a parameter, as detailed in Sect. 4-4.2, can be employed for finite integrals with a factor

$$x^K \log(x), \qquad R1(K) \neq 0,$$

in the integrand.

Examples :

```
                            K
(D1)                     LOG (X)

(C2) DEFINT(D1,X,0,1)@                                      [12]

                              K
(D2)               (-1)  GAMMA(K + 1)

(C3) LOG(X)**(1/2)/(X**2)@

                       SQRT(LOG(X))
(D3)                   ------------
                             2
                            X

(C4) DEFINT(D3,X,1,INF)@                                    [12]

                        SQRT(%PI)
(D4)                    ---------
                            2

(C5) X**(1/3)*(-LOG(X))**(-1/2)@

                            1/3
                           X
(D5)               ---------------------
                   SQRT(-1) SQRT(LOG(X))

(C6) DEFINT(D5,X,0,1)@

                      SQRT(%PI)
(D6)                - ---------
                            4
                      SQRT(-)
                            3
```

```
(C7) LOG(X)*((1-X**(1/2))/X)**(1/2)@

                         SQRT(1 - SQRT(X)) LOG(X)
(D7)                     ------------------------
                                 SQRT(X)

(C8) DEFINT(D7,X,0,1)@

                               D         3 |
(D8)                         4 (--BETA(r,-))|
                               Dr        2 |r=1
```

CHAPTER VI

DESCRIPTION OF ALGORITHMS

1. Computation of Residues

1.1 Residue at a Pole

For convenience of reference the algorithm for computation of residues is repeated here.

Let $f(Z)$ be a function of a complex variable Z, analytic everywhere in a domain D except for a number of poles in D. Suppose $F(Z)$ can be written in the form

$$f(Z) = U(Z)/V(Z)$$

such that $U(Z)$ is analytic in D. This means that poles of $f(Z)$ are zeros of $V(Z)$. Suppose p is a pole of order m of $f(Z)$, the residue of $f(Z)$ at p is computed by the following algorithm.

RESIDUE ALGORITHM:

If $m = 1$, compute as the answer

$$U(p)/V'(p)$$

otherwise, if V is a polynomial,

(i) Set V to the quotient of V(Z)/(Z - p)**m which is computed by long division.

(ii) Return the result computed from

$$\frac{1}{(m-1)!}\left(\frac{d}{dZ}\right)^{m-1}\frac{U(Z)}{V(Z)}\Bigg|_{Z=p}$$

otherwise, (m > 1, V not a polynomial) compute and return as answer the limit

$$\lim_{Z\to p}\frac{1}{(m-1)!}\left(\frac{d}{dZ}\right)^{m-1}(Z-p)^{m}\,f(Z)$$

1.2 Evaluation of Contour Integrals by Residue Theory

In order to evaluate an integral of f(Z) around a closed contour C by residue theory, it is necessary to locate all poles of f inside C. After this is done, the remaining problem is to compute the sum of residues of f at these points efficiently. The difficult part is finding poles. WANDERER employs the SOLVE routine in MACSYMA to solve V(Z)=0.

For V(Z) a polynomial in Z, SOLVE finds its zeros by factoring over the integers and applying formulas to each factor of degree less than 5. For factors of higher degree only those of the form a*Z**n+b will be solved. The problem of factoring large arbitrary polynomials is non-trivial, to say the least. The development of a more powerful factoring algorithm which will factor polynomials over a larger ring than the integers would certainly be helpful to SOLVE and WANDERER. The location of zeros is usually more difficult if V(Z) is not a polynomial in Z but some more complicated function. In such a case, WANDERER usually uses methods other than the residue theory. An exception is when V(Z) is a polynomial in %E**Z which has been discussed in Sect. 4-5.2.

Thus within the limitations of SOLVE, V(Z) will be solved and its zeros sorted into a list of pairs. Each pair containing a zero and its multiplicity, such as

L = ((Z1,m1) , (Z2,m2) , (Z3,m3) , ...).

L is then sorted into two lists L1 and L2, discarding poles outside the closed contour C, such that

L1 = a list of all simple poles
L2 = a list of other poles paired with their multiplicities.

At this point, we can apply the RESIDUE ALGORITHM to obtain the desired sum. To avoid repeated calculation of V'(Z), the program checks whether L1 is empty. If L1 is not an empty list, V'(Z) will be computed and stored for possibly repeated reference later in the computation.

Note that it has been assumed from the beginning of this section that a pole of f(Z) would be a zero of V(Z). This assumption is quite reasonable for almost all of the applications in evaluation of definite integrals of elementary functions. However a zero of V(Z) need not be a pole of f(Z). For instance Z = %I is a zero of

$$Z^4 - 1$$

but not a pole of

$$\frac{(Z + \%I)}{(Z^4 - 1)}$$

For each zero p of V(Z) we may check the value of U(p) to see if p is really a pole of f(z). Although it is not clear what can be done if U(p) = 0, since p may still be a pole of lower order. A better method is to ignore the fact that if V(p) = 0, U(p) may also be 0 and pretend that p is an actual pole of f(Z). This is valid because a residue at any removable singular point will turn out to be 0. Furthermore, the residue obtained at a pole of order m is not changed if any integer $n > m$ is used as the order of p in calculating the residue.

It is conceivable that V(Z) may have an infinite number of zeros. Not being able to sum infinite series, WANDERER can not evaluate integrals which require such a computation. Sometimes only a finite number of poles are inside the closed contour. One such case WANDERER handles is when V(Z) is a polynomial in %E**Z as described in Sect. 4-5.2.

2. Obtaining Real and Imaginary Parts

In the course of evaluating a definite integral by complex contour integration, the need to take the real or the imaginary part of an expression often arise. For instance, to see whether a pole, p, lies above or below the real axis the sign of Im(p), the imaginary part of p, is examined. To determine if p lies inside the unit circle at Z=0, it is needed to compute ABS(p) which involves taking the real and imaginary parts of p.

The algorithm for obtaining Rl(p) is presented as a representative of similar procedures used.

Algorithm REALPART(p) :

1) If p is %I return 0, if p is a number or any other atomic symbol, return p.
2) If p is a sum, ($p = \sum_{j=1}^{n} pj$) then return
   $\sum_{j=1}^{n}$ REALPART(pj)
3) If p is a product (p = p1*p2) then return
   REALPART(p1)*REALPART(p2)
   - IMPART(p1)*IMPART(p2)
4) If p = %E**p1 return
   %E**REALPART(p1)*COS(IMPART(p1))

5) If p = p1**p2, convert p to %E**(p2*LOG(p1)) then go to step 4.

6) If p = LOG(p1), return LOG(ABS(p1))

7) Otherwise, return the form R1(p).

3. A Heuristic Pattern Recognition Program

It is often the case that some pattern recognition is needed, at one stage or another during the evaluation of an integral. Although many integrals can be evaluated without any pattern recognition, this capability remains important to WANDERER. One specific pattern shall be discussed as a representative of such methods in WANDERER. Consider the pattern

$$P(X) = (B\ X^{N} + A)^{M}$$

where B, N, A and M are free of X (i.e. do not involve X) and all except A must be non-zero. This pattern should match, for example, every one of the expressions

$$X, \quad 2\ X + 1, \quad (X^{1/2} + \%I)^{-1/3}$$

$$(X^{p} - 1)^{q}, \quad \text{and} \quad X^{2} + 2\ X + 1.$$

The last expression is the expansion of

$$(X + 1)^2.$$

Expressions in expanded form present some trouble for the recognition algorithm. This difficulty is overcome by the use of differentiation and rational simplification. That is if an expression E(X) is equivalent to an expression matching the pattern P(X), we can compute

$$E'(X) \;/\; E(X) = U(X) \;/\; V(X),$$

cancelling all common factors in the numerator U(X) and denominator V(X). Then V(X) should match p(X) with M = 1. As a result of this match, some values are assigned to the variables A and B. By use of these values the correct value of M and N can be recovered from U(X). The values for A and B thus obtained may differ from the true values by a constant factor. This happens whenever these true values have common factors. The real values of A and B can be determined by comparing A**M to E(0). This procedure was suggested by Moses.

The full algorithm used for matching P(X) will be described. Let it be called PM. PM uses another routine PN which recognizes the pattern

$$P(X) = (B\ X^N + A)$$

As one can see in the following algorithm, the values of B, N, A and M will be set as the matching process proceeds. If the pattern is matched, the values of these four variables are found.

Algorithm PM(E(X),X) :

1) If E = X, pattern matched. (B=1,N=1,A=0,M=1)

2) If E is free of X or involves any of the functions: SIN, COS, TAN, LOG, EXP, etc., P(X) is not matched.

3) If E is in the form r**s and s is free of X then match M to s, otherwise match M to 1. Then, if PN(r,X) succeeds in matching, the pattern is matched.

4) i) rationally simplify E'(X) / E(X) and set r to the result obtained.

ii) set s to the denominator of r, set r to the numerator of r.

iii) if PN(s,X) succeeds in matching, simplify the expression r / (N B X**(N-1)), set r to the result thus obtained and go to step 5.

iv) P(X) is not matched.

5) If r is not free of X, P(X) is not matched.

6) i) Match M to r

ii) compute and set r to E(0) / A**M

iii) If r = 1, P(X) is matched. If r≠1, first set

$$A = A r^{1/m} \text{ and } B = B r^{1/m},$$

then P(X) is matched.

Algorithm PN(E(X),X)

1) If E is free of X, the pattern not matched.

2) Match A to E(0) and set E to E - A.

3) If E is of the form r*X**s, the pattern is matched (B to r, N to s). Otherwise there is no match.

4. A Procedure for Change of Variables

Substitution of a new variable Y for a subexpression, say g(X), of the integrand in a given integral is a frequently used method in integration. Let the given integral be

$$J = \int_a^b f(X)\ dX,$$

then the transformed integral would be in the form

$$J = \int_c^d F(Y)\ dY$$

where F(Y) might be considerably simpler than f(X). F(Y), c and d are computed by a procedure which is called whenever a change of variable is needed. It makes use of two other modules of MACSYMA, namely SOLVE and DELIMITER.

i) Use SOLVE to solve for X in Y = g(X), obtaining X = h(Y), the inverse function of g.

ii) If h can not be obtained, return indication of failure.

iii) Compute, using the DELIMITER, c and d as in c = LIMIT(g(X), X, a, PLUS),

d = LIMIT(g(X), X, b, MINUS).

iv) Obtain f(Y) by assigning it a value computed from

f(h(Y)) h'(Y).

5. Solving Systems of Linear Algebraic Equations

In sect. 4-5.2 the need to compute a polynomial Q(X) from a given one P(X) satisfying a given relation

(1) Q(X)-Q(X + 2 %PI %I) = P(X)

has been mentioned.

The method of undetermined coefficients is used to determine Q(X). Let Q(X) be a polynomial in X with degree one higher than P(X) and unknown coefficients C0, C1, ... , Cn. That is

$$Q(X) = X^{n+1} + Cn\ X^{n} + \ .\ .\ .\ + C1\ X + C0.$$

Equation (1) will give n linear relations among these coefficients. Therefore the value of these C's can be obtained by solving the system of linear equations they must satisfy.

For the purpose of solving systems of linear algebraic equations, A method known as the "Two-step fraction-free Gaussian elimination" [14] has been implemented. This method is an improvement over a corresponding one-step method and features a procedure that keeps the size of the intermediate expressions in the course of the reduction down by dividing them by a common factor which the procedure can predict. The advantage of this method over a more efficient and elaborate scheme [18] is its simplicity.

6. Convergence of Integrals

There are many methods to determine the convergence or divergence of a given improper integral. Some are limit tests, others comparison tests etc. Some test for absolute convergence, others conditional or uniform convergence. It is a possible area for future work. There is no real need to have such elaborate schemes in WANDERER, for after all it is the value of the given integral that is desired. Consider

$$J = \int_a^b f(X)\, dX.$$

If f(X) is a rational function of X, WANDERER combines convergence tests with evaluation algorithms as explained in chapters 4 and 5. If f is not rational, then it is not allowed to become infinite except at a and b. This is a convention on inputs used by WANDERER. Thus, J is a proper integral if a, b, f(a) and f(b) are finite. If J is improper, WANDERER has a test for absolute divergence.

Test for absolute divergence:

(1) If a and b are finite and $b > a$, then J is absolutely divergent if

$$\text{LIMIT}((b-X)f(X),X,b,\text{MINUS}) \neq 0,$$

or

$$\text{LIMIT}((X-a)f(X),X,a,\text{PLUS}) \neq 0.$$

(2) If a is infinite, then J is absolutely divergent if

$$\text{LIMIT}(X*f(X),X,a) \neq 0.$$

(3) If b is infinite, then J is absolutely divergent if

$$\text{LIMIT}(X*f(X),X,b) \neq 0.$$

If a function G(X) exists such that G'(X) = f(X), then the method of antiderivative can be used. That is to compute J by evaluating

$$\text{LIMIT}(G(X),X,b,\text{MINUS}) - \text{LIMIT}(G(X),X,a,\text{PLUS}).$$

If this value is finite, it is the value of the given improper integral. If it does not exist, then J diverges.

CHAPTER VII
AN APPLICATION OF MACSYMA AND WANDERER

0. Introduction

The usefulness of a general purpose algebraic manipulation system such as MACSYMA in facilitating the solution of mathematical problems has been demonstrated. Using his "Symbolic Mathematical Laboratory" [15], Martin demonstrated solutions to three demanding problems in applied mathematics. These examples emphasize the fact that routine algebraic computation can be done by computer programs not only without error but much faster than by hand. The value of such a system is especially appreciated when the expressions involved are large and complicated. By employing such a computer facility the human problem solver may be freed from the tedious and uninspiring manipulations to think more about the profound aspects of his problem. But this is not all such a system can do. Moses's SIN [20], a program for indefinite integration, provides a good example of successful mechanization of a mathematical process which is far from routine or straight-forward. The introduction of SIN broadened the scope of algebraic manipulation systems significantly.

The purpose of this chapter is to show how MACSYMA and WANDERER can be used to help solve complicated problems that are of practical importance. One such problem is the asymptotic evaluation of certain contour integrals arising in mathematical physics. Usually one starts with one or a set of differential equations describing a physical problem. Solving these equations by one method or another, most often by integral transforms, one will arrive at a solution in the form of a definite integral which is often difficult, if not impossible, to evaluate exactly. Frequently, one is not so interested in the exact solution but the behavior of the system when one parameter becomes very large or small. This is where asymptotic analysis is needed.

MACSYMA is used to obtain the asymptotic solution of an infinite integral. Many facilities provided in such an algebraic manipulation system can be illustrated through this application. It is also possible to show how the definite integration capability provided by WANDERER is needed for the successful computation of the results.

1. An Asymptotic Analysis Problem and an Outline of the Method of Steepest Descent

Consider the infinite integral

$$J(C) = \int_{-\infty}^{\infty} \%E^{C\ H(t)}\ dt, \qquad C > 0$$

where

$$H(t) = -\frac{t^4}{4} - \%I\ t$$

While integrating J exactly may be impossible, its asymptotic behavior as C becomes very large can be investigated. To obtain the asymptotic expansion of J, the method of steepest descent [5,7] will be employed. Basically, the method of steepest descent consists in deforming the contour of integration in such a way that the major contribution to the integral arises from a small portion of the new path of integration. The contribution will become more and more dominant as the parameter of interest grows. This parameter here is C.

The first step in this method is to find the new path of integration. On a given contour, larger contributions comes from portions where the integrand is larger in absolute value and less oscillatory. Hence the requirements for a desirable path are: (a) the absolute value of the integrand becomes maximum at a point, t1 say, on the path. (b) The argument of the integrand is constant on the path near t1. The first requirement is obvious. The second is essential, for if the phase angle changes even slightly near t1, this change will be magnified by the very large factor C resulting in rapid oscillations of the integrand and therefore negating any possible contribution from the point t1. Let U(t1,t2)= Rl(H(t1+%I t2)) and V(t1,t2) = Im(H(t1+%I t2)). Let p be a point where

$$\frac{dU}{d\ t1} = \frac{dU}{d\ t2} = \frac{dV}{d\ t1} = \frac{dV}{d\ t2} = 0.$$

Then p is certainly a candidate for t1 required in (a) and (b). Such a point is called a saddle point. The choice of the name saddle point will be made clearer later. The Cauchy-Rieman conditions imply that criterion for such a point is H'(t)=0. There may be more than one such point in the complex t-plane. There are an infinite number of different curves which pass through a saddle point and satisfy (a) and (b). Among them the path along which the integrand decrease in size most rapidly is the best. For the integral J, this means a curve on which U decreases most rapidly.

Let $H'(b) = 0$. It can be shown by use of properties of analytic functions that U varies most rapidly on curves $Im(H(t))$=const. (usually called level curves). If $H''(b) \neq 0$, two level curves will pass through the point b, intersecting at right angle, as shown in fig. 1.

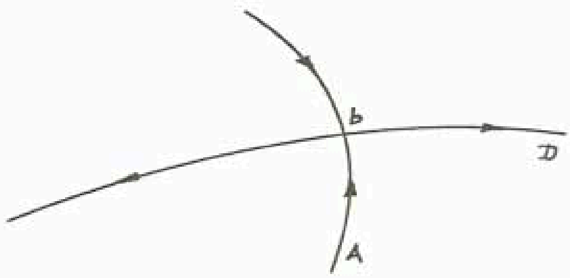


fig. 1

On one of these two curves, say curve A, $Rl(H(t))$ is minimum at b and increases as t moves along A away from b. On the other hand, $Rl(H(t))$ is maximum at t=b on the curve D and decreases as t moves away from b on D. Curve A is called the steepest ascent path and D the steepest descent path. The point b is usually referred to as a saddle point. If $H''(b) \neq 0$, b is a saddle point of order 1, if $H''(b)=0$ and $H'''(b) \neq 0$ of order 2, etc. If the original contour can be deformed onto one or a combination of such steepest descent paths, then the asymototic expansion of the given integral can be obtained by a rather routine procedure which involves: change of variable of integration, inversion of truncated power series at the saddle points and term by term integration.

Let us outline the steps of the solution procedure as follows.

1) Locate and determine the order of saddle points of H(T).

2) Compute U(t1,t2) and V(t1,t2) such that
H(t1 + %I t2) = U(t1,t2) + %I V(t1,t2).

3) Obtain V(t1,t2) = const. curves which pass through the relevant saddle points.

4) Examine the V(t1,t2) = const. curves to determine whether deformation of contour can be made to curves through the saddle points.

5) Change the variable of integration.

6) Express t as a truncated series in the new variable about each relevant saddle point.

7) Determine the coefficients in the above series.

8) Apply Watson's lemma to obtain the first few terms of the asymptotic expansion by integrating term by term.

2. Solution Steps

Presented here are the solution steps of the asymptotic expansion of J(C) in the exact sequence as they have been carried out using MACSYMA. The lines labelled (CI) are input commands. A command line ends with either a @ sign or a $ sign. The @ sign causes results obtained by executing the command line to be displayed in a subsequent line labelled (DI). A (CI) line together with a corresponding (DI) line will be referred to as step I. The $ as an end of command line character suppresses display of results for that line. Explanatory texts will be inserted between lines. To avoid becoming a user's manual for MACSYMA, explanation for the commands used in the solution will be made quite brief. For a more detailed look at MACSYMA the reader is referred to [16].

```
(C1) P:8$
```

P is a parameter which is set depending on the number of terms desired in the asymptotic expansion. By setting P to 8, we shall obtain the first 4 terms. The reason will become evident later.

```
(C2) H(T):=-T**4/4-%I*T@
```

(D2) $$H(T) := -\frac{T^4}{4} - \%I\ T$$

defining the function H(T)

(C3) DIFF(H(T),T)=0@

(D3) $$-T^3 - \%I = 0$$

creating an equation H'(T)=0

(C4) SOLVE(%,T)@
SOLUTION

(E4) $$T = \frac{SQRT(3) - \%I}{2}$$

(E5) $$T = \%I$$

(E6) $$T = \frac{-SQRT(3) - \%I}{2}$$

(D6) (E4, E5, E6)

The % sign used in (C4) stands for the last (D3), in general a % sign represents the last expression labelled (Di). SOLVE in (C4) is an invocation of the MACSYMA SOLVE program (see [10] for its capabilities and limitations). The roots of H'(T)=0 give three first order saddle points. Passing through each of these points there will be one steepest ascent and one steepest descent level curve (Im (H(t))= const. curves). The saddle points in (E4) and

(E6) are the points B and A respectively shown in fig. 2. The value of H at these three saddle points will now be computed.

```
(C7) HB:RATSIMP(H(PART(E4,2)))@

                                 - 3 %I SQRT(3) - 3
(D7)                             ------------------
                                         8
```

What has been done in step 7 is the computation and simplification of H(B). The command RATSIMP causes rational simplification which is essentially putting expressions to be simplified into the form of one numerator and one denominator and perform all possible GCD cancellations. The command PART allows a user to obtain subexpressions of an expression. PART(E4,2) returns the second part of the equation E4 which is its right-hand side. Note that commands can be nested.

```
(C8) HC:RATSIMP(H(PART(E5,2)))@

                                           3
(D8)                                       -
                                           4

(C9) HA:RATSIMP(H(PART(E6,2)))@

                                   3 %I SQRT(3) - 3
(D9)                               ----------------
                                          8
```

The point T=%I turns out to be irrelevant because the new path which will be determined does not pass through it. The next goal is to obtain curves passing through the saddle points (SQRT(3)/2, -1/2) and (-SQRT(3)/2, -1/2) along which the imaginary part of H(T) is constant and the real part of H(T) varies most rapidly, i.e. the steepest paths through the saddle points.

```
(C10) EXP:EXPAND(H(T1+%I*T2))@

              4                             2    2
            T2                3        3 T1   T2
(D10)  -  ---  +  %I T1 T2   +  ---------
             4                              2

                                                           4
                                                 3       T1
                                   -%I T1   T2 + T2 - --- -%I T1
                                                          4

(C11) V(T1,T2):="COEFF(D10,%I)@

                                                3          3
(D11)                V(T1,T2):= T1 T2   - T1   T2 - T1
```

By steps 10 and 11, the imaginary part of H(T1+ %I T2) is found and given a function name V(T1,T2). The command COEFF(exp,var,n) computes the coefficient of var**n in exp. In this case, the coefficient of %I in (D10) is exactly the imaginary part of (D10).

```
(C12) U(T1,T2):="COEFF(EXP,%I,0)@
```

(D12) $$U(T1,T2) := -\frac{T2^4}{4} + \frac{3\ T1^2\ T2^2}{2} + T2 - \frac{T1^4}{4}$$

U(T1,T2) is defined to be the real part of H(T1 + %I T2).

(C13)V(T1,T2)=V(SQRT(3)/2, -1/2)@

(D13) $$T1\ T2^3 - T1^3\ T2 - T1 = -\frac{3\ SQRT(3)}{8}$$

Obtained in (D13) is the equation of a curve, V = constant, which passes through the saddle point B, (SQRT(3)/2, -1/2). This curve will be referred to as CR.

(C14)V(T1,T2)=V(-SQRT(3)/2, -1/2)@

(D14) $$T1\ T2^3 - T1^3\ T2 - T1 = \frac{3\ SQRT(3)}{8}$$

(D14) is the equation of a curve, V=constant, passing through the saddle point A, (-SQRT(3)/2, -1/2). Let this curve be CL. The curves CL and CR have to be examined carefully by the human problem solver to determing the new path of integration. The manner in which they extend to infinity is often important in deforming the contour. Asymptotes to these curves can be found easily. For an algebraic curve, f(X,Y)=0, the way to find asymptotes is to substitute m*X+c for Y in f(X,Y)=0 then determine values of m and c such that the equation has two infinite roots (i.e., equating to zero the coefficient of the highest and second highest powers of X). Asymptotes parallel to X=0 are missed by this method. f(X,Y)=0 has such an asymptote if it is possible to choose h in such a way that the equation f(h,Y)=0 has two infinite roots. Asymptotes will be found for CL and CR in the next two steps.

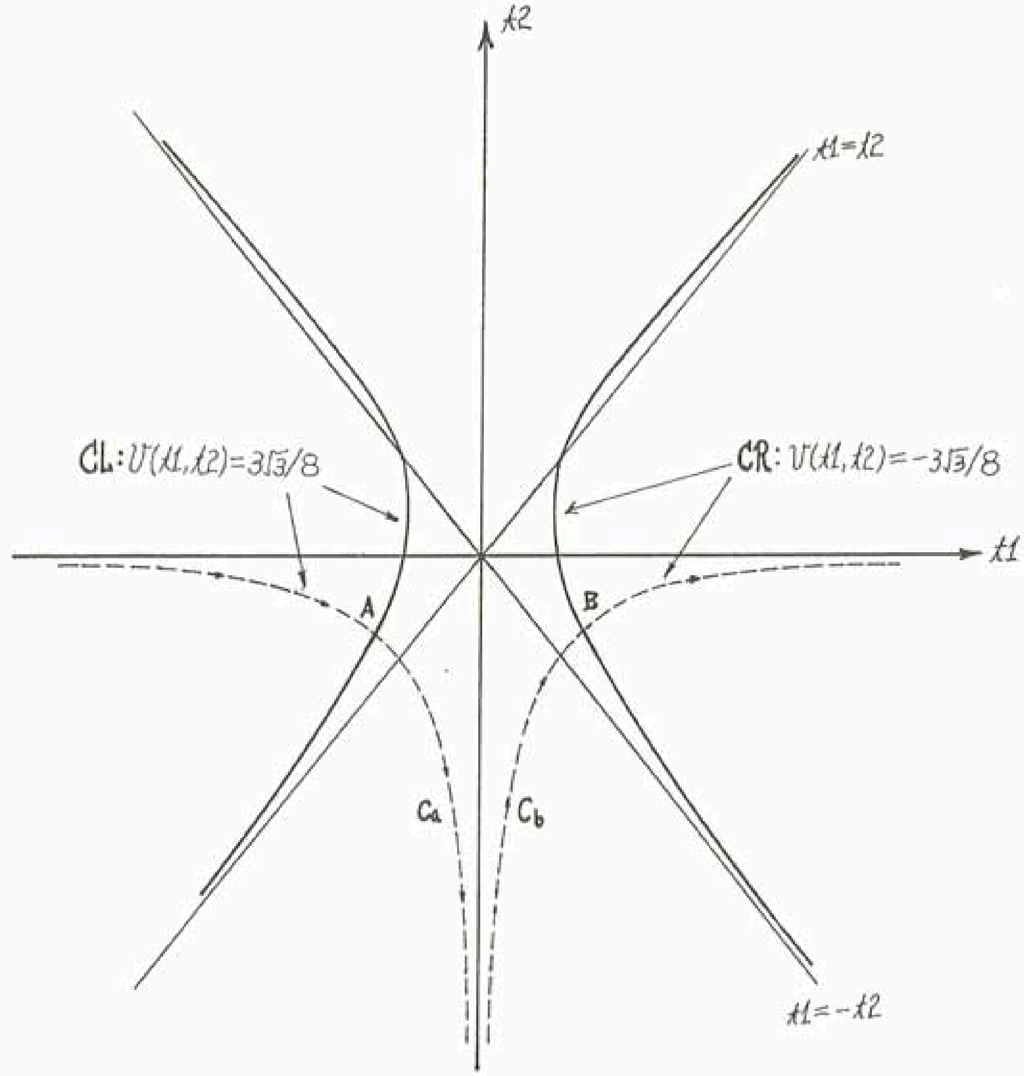
(C15) SUBSTITUTE((T2=A*T1+B),PART(D14,1))@

(D15) $$- T1^3 (A\ T1 + B) + T1\ (A\ T1 + B)^3 - T1$$

(C16) RATSIMP(%)@

(D16) $$(A^3 - A)\ T1^4 + (3\ A^2 - 1) B\ T1^3 + 3\ A\ B^2\ T1^2 + (B^3 - 1)\ T1$$

The asymptotes are clearly T1=0, T2=0, T1=T2 and T1= - T2. With the aid of MACSYMA to generate points we plotted the curves as shown in fig. 2. with steepest descent paths labelled Ca and Cb. This figure is not a computer generated plot.


λ2
λ1=λ2
CL: V(λ1, λ2)=3√3/8
CR: V(λ1, λ2)=−3√3/8
λ1
A
B
Ca
Cb
λ1=−λ2

It can be seen in fig. 2 that by use of Cauchy's integral theorem, the original contour can be deformed to the contour Ca+Cb. For the purpose of change of path of integration a new variable R is introduced which will be the parameter of our steepest descent paths.

```
(C17) -R**2=H(T)-H(T0)@
```

(D17) $$- R^2 = \%I\ T0 - \frac{T^4}{4} - \%I\ T + \frac{1}{4}$$

T0 stands for either of the two saddle points A or B. Indeed if T0=A, R is real if and only if T is a point on Ca, for Im(H(T)-H(A))=0 and 0 $\geq$ (H(T)-H(A)) only for points on Ca. The same can be said about T0=B. T has to be expressed in terms of R in order to perform the change of variable. Solving for T as a function of R exactly in (D17) is not necessary. What is needed is the first few terms of a power series expansion of T in terms of R about the point T=T0 (the saddle point).

```
(C18)T=T0+DOSUM(J,1,P,B[J]*R**J)@
```

(D18) $$T = T0 + B_8 R^8 + B_7 R^7 + B_6 R^6 + B_5 R^5 + B_4 R^4 + B_3 R^3 + B_2 R^2 + B_1 R$$

T is set to a truncated power series in R. The unknown coefficients BI will be solved for by use of (D17).

```
(C19) E:SUBSTITUTE(D18, R**2+PART(D17,2))@
```

$$\text{(D19)} \quad -(T0 + B_8 R^8 + B_7 R^7 + B_6 R^6 + B_5 R^5 + B_4 R^4 + B_3 R^3 + B_2 R^2 + B_1 R)^4/4 - \%I\,(T0 + B_8 R^8 + B_7 R^7 + B_6 R^6 + B_5 R^5 + B_4 R^4 + B_3 R^3 + B_2 R^2 + B_1 R) + \%I\ T0 + R^2 + \frac{1}{4}$$

The expression (D19) has been given a name E. Before E is expanded in order to collect terms, a few simplification rules are defined so as to discard powers of R higher than 8 in the expansion process. This greatly reduces the intermediate expression bulge which would otherwise occur.

```
(C20) DECLARE (N,NPRED)$
(C21) NPRED(X):=IF X > P THEN TRUE ELSE FALSE$
(C22) TELLSIMP (R**N,0)$
0
IS THE REPLACEMENT
```

A simplification rule has been set up so that any R**n is replaced by 0 if N > p which is 8.

```
(C23) TELLSIMP(T0**3,-%I)$
```

An additional simplification rule is defined so that T0**3 shall be replaced by -%I automatically.

```
(C24) RATVARS(R)$
```

R shall be the main variable in subsequent rational simplifications.

```
(C25) E:PART(RAT(E),1)@
```

(D25)

$$
\begin{aligned}
& R^{8} ((- 12\, B_{1}\, B_{7} - 12\, B_{2}\, B_{6} - 12\, B_{3}\, B_{5} - 6\, B_{4}^{2})\, TO^{2} \\
& \quad + (- 12\, B_{1}^{2}\, B_{6} - 24\, B_{1}\, B_{2}\, B_{5} + (- 24\, B_{1}\, B_{3} - 12\, B_{2}^{2})\, B_{4} \\
& \quad - 12\, B_{2}\, B_{3}^{2})\, TO - 4\, B_{1}^{3}\, B_{5} - 12\, B_{1}^{2}\, B_{2}\, B_{4} - 6\, B_{1}^{2}\, B_{3}^{2} \\
& \quad - 12\, B_{1}\, B_{2}^{2}\, B_{3} - B_{2}^{4}) \\
& + R^{7} ((- 12\, B_{1}\, B_{6} - 12\, B_{2}\, B_{5} - 12\, B_{3}\, B_{4})\, TO^{2} \\
& \quad + (- 12\, B_{1}^{2}\, B_{5} - 24\, B_{1}\, B_{2}\, B_{4} - 12\, B_{1}\, B_{3}^{2} - 12\, B_{2}^{2}\, B_{3})\, TO \\
& \quad - 4\, B_{1}^{3}\, B_{4} - 12\, B_{1}^{2}\, B_{2}\, B_{3} - 4\, B_{1}\, B_{2}^{3}) \\
& + R^{6} ((- 12\, B_{1}\, B_{5} - 12\, B_{2}\, B_{4} - 6\, B_{3}^{2})\, TO^{2} \\
& \quad + (- 12\, B_{1}^{2}\, B_{4} - 24\, B_{1}\, B_{2}\, B_{3} - 4\, B_{2}^{3})\, TO \\
& \qquad - 4\, B_{1}^{3}\, B_{3} - 6\, B_{1}^{2}\, B_{2}^{2}) \\
& + R^{5} ((- 12\, B_{1}\, B_{4} - 12\, B_{2}\, B_{3})\, TO^{2}
\end{aligned}
$$

$$+ (-12 B_1^2 B_3 - 12 B_1 B_2^2) TO - 4 B_1^3 B_2)$$

$$+ R^4 ((-12 B_1 B_3 - 6 B_2^2) TO^2 - 12 B_1^2 B_2 TO - B_1^4)$$

$$+ R^3 (-12 B_1 B_2 TO^2 - 4 B_1^3 TO) + R^2 (4 - 6 B_1^2 TO^2)$$

E is set to the huge expression above which is the numerator of a truncated expansion of E; the denominator is a constant.

```
(C26)FOR J:1 STEP 1 UNTIL J > (P-1)
     DO BLOCK (EQ[J]:COEFF(E,R,J+1)=0,DISPLAY(EQ[J]))$
```

This do loop is used to generated the coefficients of the various powers of R in E and store them in the form of equations in an array RL. Each EQ[J] will be displayed after it is set.

$$4 - 6 B_1^2 TO^2 = 0$$

$$- 12 B_1 B_2 TO^2 - 4 B_1^3 TO = 0$$

$$(-12 B_1 B_3 - 6 B_2^2) TO^2 - 12 B_1^2 B_2 TO - B_1^4 = 0$$

$$(-12 B_1 B_4 - 12 B_2 B_3) TO^2$$

$$+ (-12 B_1^2 B_3 - 12 B_1 B_2^2) TO - 4 B_1^3 B_2 = 0$$

$$(-12 B_1 B_5 - 12 B_2 B_4 - 6 B_3^2) TO^2$$

$$+ (-12 B_1^2 B_4 - 24 B_1 B_2 B_3 - 4 B_2^3) TO$$

$$- 4 B_1^3 B_3 - 6 B_1^2 B_2^2 = 0$$

$$(-12 B_1 B_6 - 12 B_2 B_5 - 12 B_3 B_4) TO^2$$

$$+ (-12 B_1^2 B_5 - 24 B_1 B_2 B_4 - 12 B_1 B_3^2 - 12 B_2^2 B_3)$$

$$TO - 4 B_1^3 B_4 - 12 B_1^2 B_2 B_3 - 4 B_1 B_2^3 = 0$$

$$(-12 B_1 B_7 - 12 B_2 B_6 - 12 B_3 B_5 - 6 B_4^2) TO^2$$

$$+ (-12 B_1^2 B_6 - 24 B_1 B_2 B_5 + (-24 B_1 B_3 - 12 B_2^2)$$

$$B_4 - 12 B_2 B_3^2) TO - 4 B_1^3 B_5 - 12 B_1^2 B_2 B_4$$

$$- 6 B_1^2 B_3^2 - 12 B_1 B_2^2 B_3 - B_2^4 = 0$$

Being conscious of the lack of storage space left for subsequent manipulations, we get rid of the computation history to create some space.

```
(C27) KILL(HISTORY)@
(C1) SOLVE(EQ[1],B[1])@
```

(E1) $$B_1 = \frac{SQRT(6)}{3\ TO}$$

(E2) $$B_1 = - \frac{SQRT(6)}{3\ TO}$$

(D2) (E1, E2)

The choice of the value for B1 here affects only the sense with which the steepest decent paths are traversed. Thus if we choose (E1), R would vary from INF to MINF on Ca and from MINF to INF on Cb.

```
(C3) B[1]:PART(E1,2)$
```

A do loop is used in (C4) to solve for the remaining B's (B2 through B7).

```
(C4) FOR J:2 STEP 1 UNTIL J > (P-1)
     DO BLOCK (SOL:EV(SOLVE(EQ[J],B[J]),EVAL),
          B[J]:PART(SOL,2),DISPLAY(SOL))$
```

The EV command, with EVAL as the third argument, causes the answer returned by SOLVE to be evaluated once more. Since every Bj is solved in terms of earlier B's which already have some value, one more level of evaluation will eliminate the dependencies on previous B's.

$$B_2 = - \frac{2 \%I}{9}$$

$$B_3 = \frac{10 \%I - 3 TO}{27 \, SQRT(6) \, TO^2}$$

$$B_4 = \frac{9 \%I \, TO + 23}{243 \, TO}$$

$$B_5 = \frac{- 3 \%I \, TO^2 + 28 \, TO - 108 \%I}{972 \, SQRT(6)}$$

$$B_6 = \frac{225 \%I \, TO^2 - 342 \, TO + 1031 \%I}{26244 \, TO^2}$$

$$B_7 = - \frac{5 \%I \, TO^2}{1458 \, SQRT(6)} + \frac{SQRT(6) \%I \, TO^2}{729} + \frac{47 \, SQRT(6) \, TO}{6561} + \frac{107 \, TO}{2187 \, SQRT(6)} - \frac{6839 \%I}{52488 \, SQRT(6)} - \frac{170 \, SQRT(6) \%I}{19683} + \frac{1927 \%I}{26244 \, SQRT(6)}$$

The unknown coefficients BI have been determined. The next goal is to compute

$$\int_{-\infty}^{\infty} EXP(-C\ R^{2})\ \frac{dT}{dR}\ dR$$

for T0=A and T0=B. Odd terms of R in $\frac{dT}{dR}$ do not contribute. Therefore only 4 terms need be integrated. These terms are even in R, thus the range of integration can be changed to (0 INF). A function FN(X) will be defined for carrying out this term-by-term integration at a variable point T0=X.

```
(C5) FN(X):=BLOCK(ANS:0, FOR J:1 STEP 2 UNTIL J > (P-1)
     DO ANS:ANS+J*RATSIMP(EV(B|J|,T0=X))*
 DEFINT(%E**(-C*R**2)*R**(J-1),R,0,INF),RETURN(ANS))$
(C6) FN(-SQRT(3)/2-%I/2)@
IS THE EXPRESSION
 - C
POSITIVE, NEGATIVE, OR ZERO
NEGATIVE@
```

For the purpose of integration, WANDERER askes the user about the sign of C. The contribution from saddle point A is given in the next result.

```
                 SQRT(6) SQRT(%PI)
(D6)    -   --------------------------
            (3 SQRT(3) + 3 %I) SQRT(C)
```

```
  3 SQRT(%PI) (3 SQRT(3) + 23 %I)
+ ------------------------------------
                                     3/2
  4 SQRT(6) (27 %I SQRT(3) + 27) C
  5 SQRT(%PI) (25 SQRT(3) + 247 %I)
- -----------------------------------
                           5/2
           5184 SQRT(6) C
- (35 SQRT(%PI) (SQRT(6) (672 SQRT(3) + 1816 %I)

      3582 SQRT(3)    13077 %I                7/2
    - ------------ - --------))/(839808 C    )
        SQRT(6)       SQRT(6)

(C7) FN(SQRT(3)/2-%I/2)@

IS THE EXPRESSION
 - C
POSITIVE, NEGATIVE, OR ZERO

NEGATIVE@

           SQRT(6) SQRT(%PI)
(D7)  --------------------------
      (3 SQRT(3) - 3 %I) SQRT(C)
                 3 SQRT(%PI) (3 SQRT(3) - 23 %I)
               + ------------------------------------
                                                    3/2
                 4 SQRT(6) (27 %I SQRT(3) - 27) C
                 5 SQRT(%PI) (25 SQRT(3) - 247 %I)
               + -----------------------------------
                                          5/2
                          5184 SQRT(6) C

               + (35 SQRT(%PI) (SQRT(6) (672 SQRT(3) - 1816 %I)

                     3582 SQRT(3)    13077 %I                7/2
                   + ------------ - --------))/(839808 C    )
                       SQRT(6)       SQRT(6)
```

The above is the contribution from saddle point B. **Now** the final result is obtained by computing the following

$$2\ (\%E^{(-3\ \sqrt{3}\ \%I-3)/8}\ D7\ -\ 2\ \%E^{(3\ \sqrt{3}\ \%I-3)/8}\ D6)$$

where D6 and D7 are as given above. This expression is, by inspection, equivalent to

$$2\ (\%E^{(-3\ \sqrt{3}\ \%I-3)/8}\ D7\ +\ \%E^{(3\ \sqrt{3}\ \%I-3)/8}\ \overline{D7})$$

where $\overline{D7}$ is the complex conjugate of D7. Therefore,

```
(C8) 4*RL(%E**HA*D7)@
               3 SQRT(3) %I - 3
               ----------------
                      8                SQRT(6) SQRT(%PI)
(D8)   4 RL(%E                    (--------------------------
                                   (3 SQRT(3) - 3 %I) SQRT(C)

                3 SQRT(%PI) (3 SQRT(3) - 23 %I)
              + ----------------------------------
                                                3/2
                4 SQRT(6) (27 %I SQRT(3) - 27) C

                5 SQRT(%PI) (25 SQRT(3) - 247 %I)
              + ----------------------------------
                                    5/2
                        5184 SQRT(6) C

              + (35 SQRT(%PI) (SQRT(6) (672 SQRT(3) - 1816 %I)

                   3582 SQRT(3)    13077 %I               7/2
                 + ------------ - --------))/(839808 C     ))
                     SQRT(6)       SQRT(6)
```

This is the first four terms of the desired asymptotic expansion. RL is not a command of MACSYMA. It is used here to denote the real part of an expression.

CHAPTER VIII

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The evaluation of definite integrals is a classical problem in mathematics. Great ingenuity is frequently required, with many integrals demanding special devices. The lack of a sufficiently general theory makes evaluation of definite integrals very difficult. It is doubtful that a theory, comparable in generality to the Risch integration algorithm [26] for indefinite integrals, can be developed in the near future. We have shown that the convergence of a class of integrals of elementary functions is recursively undecidable. The proof is in appendix E. The WANDERER program presented here is a prototype heuristic computer program for the symbolic evaluation of definite integrals. It contains both general methods such as contour integration, residue theory and differentiation with respect to a parameter, in addition to quite a few special methods for specific types of integrals. Clues as to which method to use for a given problem are obtained from the integration range and the form of the integrand. Although many types of definite integrals can be evaluated by WANDERER, it, as almost any other heuristic computer program, has its limitations. Foremost among these is the fact that it is a program for the evaluation of real definite integrals of elementary functions.

It is hoped that the work reported here may provide a starting point for new approaches to the evaluation of definite integrals from the viewpoint of symbol manipulation. The advantage of this approach is twofold: (1) A computer can use integration methods that are too lengthy or complicated to be carried out by hand; (2) Such a computer program contains a collection of powerful methods that can interact with one another and can produce answers to integrals not present in any finite table. For this reason, a good definite integration program together with other facilities provided in an algebraic manipulation system can be very useful to applied mathematicians.

DELIMITER is a rather sophisticated program for computing limits of elementary functions. Such a program has been shown to be a very useful tool in an algebraic manipulation system. The method of comparing orders of infinity used in DELIMITER is an important concept which is useful in places other than the computation of limits.

A basic assumption of DELIMITER is that the operator LIMIT is distributive over the operators +, -, *, and **, i.e., rules such as the limit of a sum is the sum of the limits hold. These rules are valid only if the subproblems thus generated produce answers which do not lead to an indeterminate form. Of course there is L'Hospital's rule which can be applied to some indeterminate forms, but generally only those involving only the operator * or **. Although many aspects of the indeterminate form (INF-INF) have been considered in chapter 2, the algorithm for its determination is not complete. A powerful series expansion program would be helpful in some cases. Yet, such a program can not solve all problems. Consider for instance, the limit problem

$$\mathrm{LIMIT}(\mathrm{SIN}^2\,X + \mathrm{COS}^2\,X, X, \mathrm{INF}).$$

To obtain such limits, an algebraic manipulation system must be able to detect all constant identities. This is not possible for the set of all expressions [22]. It may be possible for a proper subset of all expressions. For example, many trigonometrical identities disappear if all trigonometric functions are radically transformed into sums of complex exponentials. This is a basic problem of great practical and theoretical interest [6]. Such simplification capabilities would be of use to many other programs as well.

DELIMITER can be augmented by increasing the types of functions it can handle which may include functions defined by integrals. Another possible area of research is the automatic determination of superior and inferior limits.

The fact that WANDERER can evaluate many non-trivial definite integrals does not mean that it can compete with an expert human integrator yet. For one thing, a mathematician can usually construct a function of a complex variable and a suitable contour for evaluating different integrals. WANDERER cannot form a contour based on analysis of a given real integral. It simply selects from the cases known to it. WANDERER would be much more powerful if it could determine, for a given integral, whether the method of contour integration and residue theory were applicable, and, if it were, evaluate the integral by forming a suitable integral around a closed contour.

The evaluation of contour integrals by residue theory usually requires the solution of algebraic or transcendental equations. WANDERER uses the SOLVE program of MACSYMA for this purpose. SOLVE has its limitations and its improvement is also a possible area for further work.

Suppose one wants to evaluate a closed contour integral by residue theory. An interesting general question is: what knowledge about the integrand or the functions used in forming it is necessary. We think the following are essential:

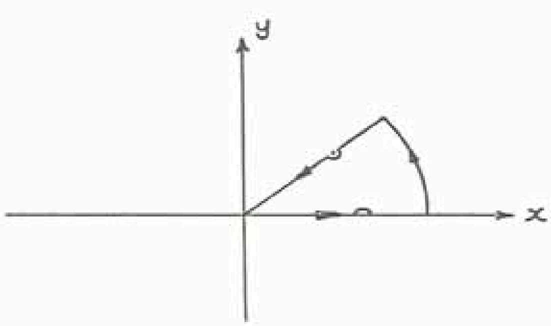
(1) evaluation;

(2) differentiability and derivatives;

(3) singularities;

(4) asymptotic behaviour.

A natural extension of the work here is integration over arbitrary user-specified contours. This should not be difficult to do, except for the lack of notation. Specifically, we must specify an arbitrary contour to a computer and devise a general data structure for use in representing contours. For instance, consider the specification of the following indented contour.



One obvious way is to specify a contour piecewise. Each piece of curve has a parametric form and a starting and ending value for the parameter. There may be other approaches.

Further work in this area of symbol manipulation might include:

(1) Design of computer algorithms for testing convergence and divergence of integrals,

(2) Summation of infinite series by residue computations,

(3) Investigation of algorithms for definite integration of special functions.

(4) Evaluation of multiple definite integrals.

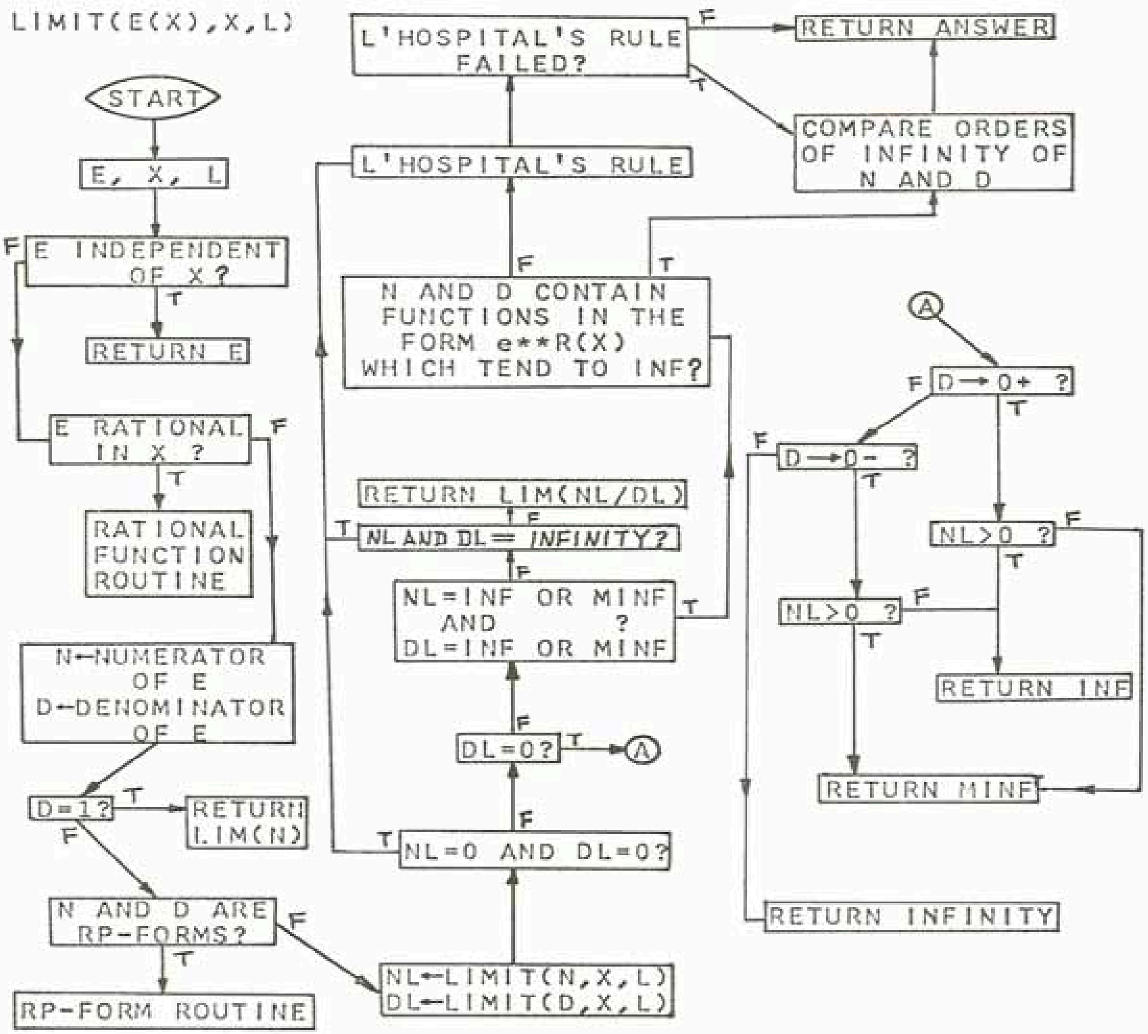
(5) Application of definite integration programs in solution of differential equations.

APPENDIX A

A FLOWCHART OF DELIMITER

A flowchart is presented in the next page which details the flow of control of DELIMITER. Listings of the programs, written in the LISP programming language, may be obtained from the author.

The routine LIM, appearing in the flowchart, is a program which applies the 'trivial' limit rules, makes use of subroutines to compute limits of sums, products, powers, and the functions SIN, COS, TAN, LOG, SINH, COSH, TANH. It calls LIMIT recursively and makes use of L'Hospital's rule and other routines when needed.



LIMIT(E(X),X,L)
START
E, X, L
E INDEPENDENT OF X?
RETURN E
E RATIONAL IN X ?
RATIONAL FUNCTION ROUTINE
N←NUMERATOR OF E
D←DENOMINATOR OF E
D=1?
RETURN LIM(N)
N AND D ARE RP-FORMS?
RP-FORM ROUTINE
NL←LIMIT(N,X,L)
DL←LIMIT(D,X,L)
NL=0 AND DL=0?
DL=0?
NL=INF OR MINF AND DL=INF OR MINF ?
NL AND DL = INFINITY?
RETURN LIM(NL/DL)
N AND D CONTAIN FUNCTIONS IN THE FORM e**R(X) WHICH TEND TO INF?
L'HOSPITAL'S RULE
L'HOSPITAL'S RULE FAILED?
RETURN ANSWER
COMPARE ORDERS OF INFINITY OF N AND D
A
D→0+ ?
D→0- ?
NL>0 ?
NL>0 ?
RETURN INF
RETURN MINF
RETURN INFINITY
T
F

APPENDIX B

A PROOF

Infinite integrals of

$$f(Z) = \%E^{\%I\ k\ Z^{n}}\ Z^{m}$$

where $n > 0$, $k \neq 0$ real, $Rl(m) > -1$, and $n - Rl(m) > 1$, have been discused in chapter 4-3.2. Some results derived there depends on the proof of (1) and (2) below.

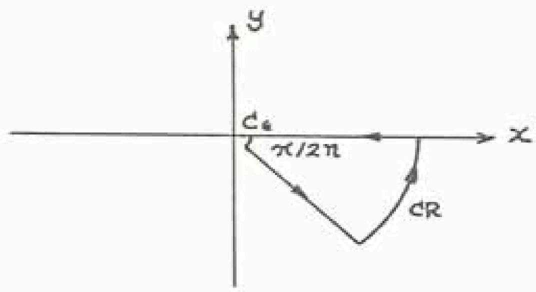
The objective here is to supply the proof of

$$1)\quad \underset{\epsilon \to 0+}{\text{limit}} \int_{C_\epsilon} f(Z)\ dZ = 0,$$

and

$$2)\quad \underset{R \to +\infty}{\text{limit}} \int_{CR} f(Z)\ dZ = 0.$$

where CR and C are the circular contours given below.



Let

$$Z = r\, e^{i\theta} \quad \text{and} \quad m = a + ib,$$

then

$$\left| \int_C f(Z)\, dZ \right| =$$

$$\left| i\, r^{m+1} \int_0^{\pi/2n} \mathrm{EXP}(i\, k r^{n} e^{i n\theta} + i\,(m+1)\,\theta)\, d\theta \right|$$

(i) $$\leq r^{a+1} \int_0^{\pi/2n} \mathrm{EXP}(-\, k\, \mathrm{SIN}(n\,\theta)\, r^{n} - b\,\theta)\, d\theta = M$$

For $0 \leq \theta \leq$ %PI/(2 n) we have

$$\mathrm{EXP}(-\, k\, \mathrm{SIN}(n\,\theta)\, r^{n}) \leq 1.$$

Thus

$$M \leq r^{a+1} \int_0^{\pi/2n} e^{-b\theta}\, d\theta$$

Since $a + 1 > 0$, this completes the proof of (1).

Now from equation (i) we have

$$M = \frac{r^{a+1}}{n} \int_0^{\pi/2} \mathrm{EXP}(-\, k\, \mathrm{SIN}(\theta)\, r^{n} - \frac{b\,\theta}{n})\, d\theta$$

$$\leq \frac{r^{a+1}}{n} \int_0^{\pi/2} \mathrm{EXP}(-\, \frac{2\, k\, \theta\, r^{n}}{\%PI} - \frac{b\,\theta}{n})\, d\theta$$

$$\leq \frac{\%PI\ r^{a+1}}{2\ k\ n\ r^{n} + \%PI\ b}(1 - EXP(-\ k\ r^{n} - \frac{\%PI\ b}{2\ n}))$$

The fact that $n-a-1 > 0$ completes the proof of (2).

APPENDIX C

PERFORMANCE OF WANDERER AND DELIMITER

As a measure of the performance of WANDERER and the limit programs, selected problems have been timed. The time sharing system used is the ITS of the Artificial Intelligence Laboratory at MIT which uses a PDP-10 computer with a memory cycle time of about 2.75 microseconds. Time used for parsing the input string and display of the computed result has been excluded in order to obtain an approximation to the time actually spent inside WANDERER or DELIMITER. Garbage collection in the LISP [17] system in which MACSYMA is written may take place during a computation. Although garbage collection is a slow procedure, it is only fair to regard it as part of the computation process being carried out. Therefore, the time it required has been included in the timing experiments. The results are put in the form of two tables. An * is used in the tables to indicate computation requiring LISP garbage collection.

TABLE I

PERFORMANCE OF DELIMITER

Limit Problem computed	Result equivalent to	Time in Sec.
$\lim_{X \to 0+} X^{X}$	1	0.82
$\lim_{X \to +\infty} (1+1/X)^{X}$	%E	1.42
$\lim_{X \to +\infty} \dfrac{X \, LOG(X) + \%E^{X}}{LOG(X^{4} + X + 1) + \%E^{SQRT(X^{3} + 1)}}$	0	3.00
$\lim_{X \to +\infty} \%E^{X \, SQRT(X^{2} + 1)} - \%E^{X^{2}}$	INF	13.18
$\lim_{X \to 2-} \dfrac{1}{X^{3} - 6 X^{2} + 11 X - 6}$	INF	1.15
$\lim_{X \to \pi/2} \dfrac{1 - SIN(X)}{COS(X)}$	0	1.52

$\lim_{X \to +\infty} \dfrac{X\ SQRT(X + 5) + 1}{SQRT(4\ X^{3} + 1) + X}$	$\dfrac{1}{SQRT(4)}$	1.93
$\lim_{X \to \frac{\pi}{2}^-} \dfrac{TAN(X)}{LOG(COS(X))}$	MINF	3.63
$\lim_{X \to 2} \dfrac{(SQRT(2\ X) - 2)}{LOG(X - 1)}$	1/2	3.83

TABLE II

PERFORMANCE OF WANDERER

Integral computed	Result equivalent to	Time in Sec.
$\int_{-1}^{1} \frac{1}{X} \, dX$	0 (principal)	0.55
$\int_{0}^{\infty} \frac{dX}{X^2 + 3X + 2}$	LOG(2)	2.02
$\int_{-\infty}^{\infty} \frac{X^2 + A X + B}{X^4 + 10 X^2 + 9} \, dX$	$\frac{\%PI\ B + 3\ \%PI}{12}$	7.42*
$\int_{0}^{\infty} \frac{X^2 + A X + B}{X^4 + 10 X^2 + 9} \, dX$	$\frac{3\ A\ LOG(3) + (B + 3)\%PI}{24}$	11.32*
$\int_{0}^{2\pi} COS^2(X) - SIN(X) \, dX$	%PI	1.67
$\int_{0}^{\infty} \frac{dX}{X^{20} + 1}$	$\frac{\%PI}{20\ SIN(\%PI/20)}$	1.08

$\int_0^\infty X^3 \, \%E^{D - 5 \, SQRT(X)} \, dX$	$(2016 \, \%E^D) / 78125$	0.95
$\int_{-\infty}^\infty \frac{COS^2(X)}{X^2 + 49} \, dX$	%PI/28	1.39
$\int_0^\infty \frac{dX}{SQRT(X) \, (2 \, X + 1)}$	%PI/SQRT(2)	0.81
$\int_0^\infty \frac{X^{1/3} \, dX}{(4 \, SQRT(X) + 3)^{3/2}}$	DIVERGENT	6.62

APPENDIX D

NOTATIONS

INF	plus infinity,	MINF	minus infinity,
%I	$\sqrt{-1}$	%PI	π
%E	base of natural logarithm		
*	multiplication operator		
**	exponentiation operator		
SQRT	square root operator		
PLOG(X)	%PI ≥ Im(PLOG(X))> -%PI, the principal branch		
LOG(X)	when X > 0 for PLOG(X)		
GLOG(X)	2 %PI > Im(GLOG(X)) 0		
ZP	the zero parameter (defined in chapter 4)		
BETA	the beta function		
GAMMA	the gamma function		

APPENDIX E

AN UNDECIDABILITY RESULT

Let $S1 = \{P(X1, X2, \ldots, Xn)\}$ be a set of polynomials with integral coefficients in $X1, \ldots, Xn$ and $S2$ a set of functions F of the form

$$F(X1, \ldots, Xn)$$
$$= (n+1)^2 \left(P^2(X1, \ldots, Xn) + \sum_{j=1}^{n} (\mathrm{SIN}^2 \pi Xj) Kj^2 (X1, \ldots, Xn)\right) - 1,$$

where Kj is the dominating function |6| for

$$\frac{d}{dXj} P^2.$$

Richardson has shown |22,6| that

Lemma 1. For F in $S2$ the predicate "there exists an n-tuple B of real numbers such that $F(B) < 0$" is recursively undecidable.

Lemma 2. If $F(B) < 0$ for some n-tuple B real numbers then there exists an n-tuple A of nonnegative integers such that $P(A) = 0$ and therefore $F(A) = -1$.

Lemma 3. Let

$$h(X) = X\ \mathrm{SIN}(X) \text{ and } g(X) = X\ \mathrm{SIN}(X^3)$$

Then for any real numbers A1, ... , An and any $0 < \epsilon < 1$ there exists $b > 0$ such that
$|h(b)-A1|<\epsilon$, $|h(g(b))-A2|<\epsilon$, ... , $|h(g(...g(b)...))-An|<\epsilon$.

By use of these Lemmas we can show

Corollary. For any G in a set of functions of the form
$G(X) = F(h(X),h(g(X)), \ldots , h(g(\ldots(g(X))\ldots))) + 1/2$,
the predicate "there exists a real number t such that G(t) = 0" is recursively undecidable.

<u>Proof</u>: Suppose the predicate is recursively decidable, then we have
1) If there exists a t such that G(t)=0 then there exist real numbers A1, ... , An such that F(A1, ... , An) < 0;
2) If there exist real numbers A1, ... , An such that F(A1, ... , An) < 0 then (by lemma 2) there exist nonnegative integers B1, ... , Bn such that F(B1, ... , Bn)=-1. Thus, there exists a real number c such that G(c) < 0. Since G(X) can be large and positive and G(X) is continuous, this implies that there exist a t such

that G(t)=0.

This contradicts lemma 1.

From the definition of the Function F, one can see G is always -1/2 and G is large and positive except at the vicinity of a finite number of points where G is negative.

<u>Theorem</u>: The convergence of a set of integrals of the form

$$\int_{-\infty}^{\infty} \frac{dX}{(X^2+1)G^2(X)}$$

is recursively undecidable.

Proof: This integral is convergent if and only if G(X) has no real zero.

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